GEORGIA INSTITUTE OF TECHNOLOGY
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SPONSORED PROJECT INITIATION

Ex-5

Date: 8/21/78

Project Title: Control of Time-Delay Systems Using Continuous-Time and Sampled-Data Models

Gr Cd

Project No: E-21-633

Project Director: Dr. Arild Thowsen

Sponsor: National Science Foundation

Agreement Period: From 9/1/78 Until 2/28/81
(Grant Period -- includes flexibility period)

Type Agreement: Grant No. ENG78-12231

Amount:
$63,632 NSF
5,384 GIT (E-21-328)
$69,016 TOTAL

Reports Required: Annual Progress Report; Final Project Report

Sponsor Contact Person (s):

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Defense Priority Rating: N/A

Assigned to: Electrical Engineering (School/Laboratory)

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GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION

SPONSORED PROJECT TERMINATION

Date: January 20, 1981

Project Title: Control of Time-Delay Systems Using Continuous-Time and Sampled-Data Models

Project No: E-21-633

Project Director: Dr. Arild Thowsen

Sponsor: National Science Foundation

Effective Termination Date: 2/28/81 → 1/16/81 Final Report submitted

Clearance of Accounting Charges: - - - -

Grant/Contract Closeout Actions Remaining:

- _ Final Invoice and Closing Documents
- X Final Fiscal ~~Report~~ Accounting (FCTR)
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Assigned to: Electrical Engineering (School/Laboratory)

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CA-4 (1/79)

E21-633

NATIONAL SCIENCE FOUNDATION
Washington, D.C. 20550

FINAL PROJECT REPORT

NSF FORM 98A

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PART I—PROJECT IDENTIFICATION INFORMATION

1. Institution and Address	2. NSF Program	3. NSF Award Number
Georgia Institute of Technology Atlanta, Georgia 30332	Systems Theory and Operations Research	ENG78-12231
	4. Award Period From 9/1/78 To 2/28/81	5. Cumulative Award Amount $63,632.00

6. Project Title

Control of Time-Delay Systems Using Continuous-Time and Sampled-Data Models

PART II—SUMMARY OF COMPLETED PROJECT *(FOR PUBLIC USE)*

The first part of the project dealt with sampled-data control of systems with time delays. Conditions were derived for state controllability using piecewise-constant controls. In particular, conditions were derived for function space null controllability, which led to algorithms for the design of digital feedback controllers that drive the true state of the system to zero in a short time. Another phase of this work centered on the construction of finite-dimensional approximations for systems with time delays. Results were obtained which showed that the stability of the given system could be investigated in terms of a finite-dimensional approximation. Another topic studied in the first part of the project was feedback stabilization of systems with delays. New sufficiency conditions for memoryless feedback stabilization were obtained.

The second part of the project dealt with a new approach to systems with time delays based on the notions of stability and stabilizability independent of delay. It was discovered that these properties can be studied using existing results on the stability of finite-dimensional systems. As a consequence, it was possible to derive constructive techniques for the study of stability and stabilizability independent of delay. The successful conclusion of this work was a result of the discovery that stability independent of delay can be characterized in terms of "two-dimensional" stability criteria.

PART III—TECHNICAL INFORMATION *(FOR PROGRAM MANAGEMENT USES)*

1. ITEM *(Check appropriate blocks)*	NONE	ATTACHED	PREVIOUSLY FURNISHED	TO BE FURNISHED SEPARATELY TO PROGRAM: Check (✓)	Approx. Date
a. Abstracts of Theses					
b. Publication Citations					
c. Data on Scientific Collaborators					
d. Information on Inventions					
e. Technical Description of Project and Results					
f. Other *(specify)*					

2. Principal Investigator/Project Director Name *(Typed)*	3. Principal Investigator/Project Director Signature	4. Date

Preface

This report, consisting of two parts, consitutes the final report to the National Science Foundation for Grant ENG78-12231. The research reported herein was completed during the funding period, September 1, 1978 to February 28, 1981, for the research project entitled "Control of Time-Delay Systems Using Continuous-Time and Sampled-Data Models". The principal investigators for this project were:

Professor Edward W. Kamen at the School of Electrical Engineering, Georgia Institute of Technology; and

Professor Arild Thowsen (Project Director), formerly at the School of Electrical Engineering, Georgia Institute of Technology; presently with the Department of Electrical Engineering, Iowa State Univeristy.

Due to the present geographical and institutional separation of the principal investigators, the final report will consist of two parts: Part I written by Professor Thowsen and Part II written by Professor Kamen.

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Part I

Part II

1. Summary

The research proposed by Dr. Thowsen under NSF Grant ENG78-12231 was concerned with the control of linear time-invariant delay-systems. A primary goal was to study sampled-data control for time-delay systems as an effective and practical method to achieve satisfactory closed-loop system performance. The results show, in general, that such control is feasible and results pertaining to the specific design requirements for sampled-data feedback controllers were obtained. In the context of the presently maturing digital-based electro-technology, the use of digital feedback controllers take on added significance, and the sampled-data control, extensively studied under the present grant, offers an efficient way of controlling dynamic systems with delays compatibly with the advances in this modern technology. Another important feature of the research reported herein is that the infinite dimensional nature of the physical processes being modelled is retained in the mathematical model used in the research. Unfortunately this is not true for all research directions pertaining to the control of continuous time-delay systems.

The research effort reported on in this part (Part I) of the final report has resulted in several refereed journal articles and conference papers. A complete listing is included in section 5 of this report. In addition, one copy of each publication is provided in the appendix section of this report.

2. Results

This section gives a brief summary description and introduction to some of the results contained in the research publications resulting from this grant.

The main research effort centered on the study of function space null controllability and zero state control of linear dynamic systems with multiple delays described by

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i x(t-i) + Cu(t) \tag{1}$$

where $x(t) \in R^n$ and $u(t) \in R^r$. The control inputs were considered to be piecewise constant functions for consistency with the use of digital controllers for regulation. Viewing $x(t)$ as a measure of the deviation from some desired trajectory point at time t, the ability to control the state

$$x_t = \{x(t+\theta), \quad -m \leq \theta \leq 0\}$$

to the zero state becomes important. Only when the state $x_\tau = 0$ can $x(t) = 0$ for all $t \geq \tau$ be achieved with zero control input. In the ensuing research summary the terms function space null controllability and zero state controllability will be used interchangeably.

Initial effort under the current NSF grant was concerned with obtaining necessary and sufficient conditions for controlling the state of system (1) with control u restricted by

$$u(t) = u(k), \quad k \leq t < k+1 \qquad k = 0,1,2,\ldots \tag{2}$$

to the zero state. Such a condition, in integral form, was derived and reported in [R-1]*. This integral criterion was shown to imply that an algebraic condition was necessary for zero state controllability [R-2]. In particular, we have the following result:

Necessary condition for system (1) - (2) to be function space null controllable at time $t > 0$ is the existence of a positive integer p such that

$$Q_p^T A_p^i B_p = 0, \quad i = 0,1,\ldots,np-1, \tag{3}$$

* Reference numbers preceded by the letter R refer to the reference list in section 5.

with

$$A_p \triangleq \begin{bmatrix} A & & & & & & & & & & & \\ B_1 & A & & & & & & & & & & \\ B_2 & B_1 & A & & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \\ B_m & B_{m-1} & \cdot & \cdot & \cdot & B_1 & A & & & & & \\ 0 & B_m & B_{m-1} & \cdot & \cdot & \cdot & B_1 & A & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & \cdot & \cdot & \cdot & B_m & B_{m-1} & \cdot & \cdot & \cdot & \cdot & B_1 & A \end{bmatrix}_{np \times np}$$

$$B_p \triangleq \begin{bmatrix} B_1 & B_2 & B_3 & \cdot & \cdot & \cdot & B_m \\ B_2 & B_3 & \cdot & \cdot & \cdot & B_m & 0 \\ B_3 & \cdot & \cdot & \cdot & B_m & 0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & & \\ B_m & 0 & & & & & \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}_{np \times nm}$$

$$Q_p^T \triangleq \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & I \end{bmatrix}_{n \times np}$$

Condition (3) was analyzed in detail for time-delay systems of order one, two and three and canonical forms for the system matrices of zero state controllable single delay systems were presented in [R-3]. One interesting observation relating to these canonical forms is that their characteristic quasipolynomials given by $\det(\lambda I - A - Be^{-\lambda})$ were polynomials in λ only. This was shown to hold true also for systems of any finite order n in both the single delay case [R-4] and the multiple delay case [R-5]. The proofs of these general results were based on properties of entire functions and on an important lemma appearing in the works of Kappel and Zverkin.

Furthermore, the problem of designing digital feedback controllers that would make $x_t = 0$ in finite time was studied. By combining results of Henry and Zverkin on the properties of dynamic systems with finite spectra with the verification of pointwise completeness of the canonical representational forms obtained in [R-3], it was possible to develop digital feedback control algorithms for low order delay systems which bring the true state to zero in a short time [R-6]. An interesting feature of this class of controllers is the freedom of design offered by one or more nonspecified feedback gains which can be effectively used to shape the system's transient response. An example of this is included in [R-6].

Another line of research was initiated in [R-7]. The central idea here is the contention that finite dimensional system models constructed by various methods from a differential-difference equation can possess much the same stability properties as the differential-difference equation itself. Use of standard criteria for stability of finite dimensional systems can then simplify the determination of time-delay system stability. On one hand it is well known that finite dimensional approximations can yield stability results which are almost diametrically opposed to those of the approximated system, yet, on the other hand, there are strong indications that use of carefully derived finite dimensional models will result in good sufficiency criteria for asymptotic stability in the general case and even stronger conditions (i.e., both necessary and sufficient conditions) in special cases. As an example of the latter, consider the scalar delay system

$$\dot{x}(t) = ax(t) + bx(t-1) + u(t) \qquad (4)$$

with feedback control

$$u(t) = cx(t) + d\dot{x}(t) \tag{5}$$

The closed-loop system (4) - (5) is asymptotically stable if and only if the two dimensional closed-loop system obtained from (4) - (5) by the truncated series expansion

$$x(t-1) \simeq x(t) - \dot{x}(t) + \frac{1}{2}\ddot{x}(t)$$

is asymptotically stable. The results in [R-7] builds on Hayes' work on transcendental equation.

Finally, new sufficiency conditions for memoryless feedback stabilization of linear time-invariant delay-differential systems were obtained for both constant and time-dependent delays in [R-8].

3. Contributions to Engineering and Science

The results obtained under NSF Grant ENG78-12231 are of potential use in several disciplines, including engineering, physics, biology, economics, and biomedicine. For specific areas of possible applications the following articles may be consulted:

W. P. London and J. A. Yorke, "Recurrent epidemics of measles, chicken pox, and mumps I: Seasonal variation in contact rates", Amer. J. Epid., Vol. 98, 1973, pp. 453-468.

J. A. Yorke, "Selected topics in differential delay equations", in Lecture Notes in Mathematics, Vol. 243, (Springer-Verlag), 1971, pp. 16-28.

F. Hoppensteadt and P. Waltham, "A problem in the theory of epidemics I, Math. Biosci, Vol. 9, 1970, pp. 71-91. Part II. Math Biosci, Vol. 12, 1971, pp. 133-145.

K. L. Cooke and J. A. Yorke, "Equations modelling population growth, economic growth and gonorrhea epidemiology", in Ordinary Differential Equations, ed. L. Weiss, (Academic Press) 1972, pp. 35-53.

A. J. Lotka and F. R. Sharpe, "Contributions to the analysis of malaria epidemiology", Am. J. Hygine, Vol. 3, 1923, pp. 1-121.

E. B. Wilson and M. H. Burke, "The epidemic curve", Proc. Nat. Acad. Sci. U.S.A., Vol. 28, 1942, pp. 361-366.

N. K. Gupta, Modelling and optimum control of epidemics, Ph.D. dissertation, University of Alberta, Edmonton, 1972.

N. K. Gupta and R. E. Rink, "A model for communicable disease control", Proc. 24th Ann. Conf. Engineering in Medicine and Biology. Las Vegas, 1971, p. 296.

P. Waltman, "A deterministic model of the spread of infection between two populations", in Delay and Functional Differential Equations and Their Application, ed. K. Schmitt, (Academic Press) 1972, pp. 281-291.

J. E. Wilkins, "The differential-difference equation for epidemics", Bull. Math. Biophys., Vol. 7, 1945, pp. 149-150.

P. J. Wangersky and W. J. Cunningham, "Time lag in population models", Cold Spring Harbor Symposia on Qualitative Biology, Vol. 22, 1957, pp. 329-338.

P. J. Wangersky and W. J. Cunningham, "On time lags in equations of growth", Proc. Nat. Acad. Sci. U.S.A., Vol. 42, 1956, pp. 670-702.

A. J. Lotka, "Analytical note on certain rhythmic relations in organic systems", Proc. Nat. Acad. Sci. U.S.A., Vol. 6, 1920, pp. 410-415.

A. J. Lotka, Elements of Mathematical Biology, (Dover) 1956.

V. Volterra, "Variations and Fluctuations of the number of individuals in animal species living together", in Animal Ecology, ed. R. N. Chapman, (McGraw-Hill) 1931, pp. 409-448.

V. Volterra, Lecons sur la Théorie Mathématique de la Lutte Pour la Vie, (Gauthier-Villars, Paris) 1931.

P. J. Wangersky and W. J. Cunningham, "Time lag in prey-predator population models", Ecology, Vol. 38, 1957, pp. 136-139.

G. Dunkel, "Single-species model for population growth depending on past history", in Lecutre Notes in Mathematics, Vol. 60, (Springer-Verlag) 1958, pp. 92-99.

M. Kalecki, "A macrodynamic theory of business cycles", Econometrica, Vol. 3, 1935, pp. 327-344.

M. Kalecki, Studies in Economic Dynamics, (Allen and Unwin) 1943.

M. Kalecki, Theory of Economic Dynamics, (Allen and Unwin) 1954.

W. H. Ray and M. A. Soliman, "The optimal control of processes containing pure time delays I: Necessary conditions for an optimum", Chem. Eng. Sci., Vol. 25, 1970, pp. 1911-1925.

D. W. Ross, "Controller design for time-lag systems with time delays", IEEE Trans. Automatic Control, Vol. AC-16, No. 6, 1971, pp. 666-672.

H. T. Banks, Modeling and Control in the Biomedical Sciences, Lecture Notes in Biomathematics, Vol. 6, (Springer-Verlag) 1975.

H. R. Bailey and E. B. Reeve, "Mathematical models describing the distribution of I^{131}-albumin in man", J. Lab. Clin. Med., Vol. 60, 1962, pp. 923-943.

H. R. Bailey and M. Z. Williams, "Some results on the differentail-difference equation $\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - T_i)$", J. Math Anal. Appl., Vol. 15, 1966, pp. 567-587.

H. B. Smets, "On the effect of delayed neutrons in reactor dynamics", Nuclear Sci. & Engineering, Vol. 25, 1966, pp. 236-241.

J. J. Levin and J. Nohel, "On a nonlinear delay equation", J. Math. Anal. Appl., Vol. 8, 1964, pp. 31-44.

J. J. Levin and J. A. Nohel, "On a system of integro-differential equations occuring in reactor dynamics", J. Math. Mech., Vol. 9, 1960, pp. 347-368. Part II, Arch. Rational Mech. Anal., Vol. 11, 1962, pp. 210-243.

W. K. Ergen, "Kinetics of the circulating-fuel nuclear reactor", J. Appl. Phys., Vol. 25, 1956, pp. 702-711.

S. Grossberg, "Learning and energy-entropy dependence in some nonlinear functional differential equations", Bull. Amer. Math. Soc., Vol. 75, 1969, pp. 1238-1242.

S. Grossberg, "A prediction theory for some nonlinear functional differential equation. I. Learning of lists", J. Math. Anal. Appl., Vol. 21, 1968, pp. 643-694. "II. Learning of patterns", J. Math. Anal. Appl., Vol. 22, 1968, pp. 422-490.

H. T. Milhorn, Jr., R. Benton, R. Ross, and A. C. Guyton, "A mathematical model of the human respiratory control system", Biophysical Journal, Vol. 5, 1965, pp. 27-46.

F. S. Grodins, J. Buell, and A. J. Bart, "Mathematical analysis and digital simulation of the respiratory control system", J. Appl. Physiology, Vol. 22, 1967, pp. 260-276.

S. B. Norkin, Differential Equations of the Second Order with Retarded Argument, Translation of Mathematical Monographs, Vol. 31, American Math Soc., 1972.

K. J. Arrow, T. Harris, and J. Marschak, "Optimal inventory policy", Econometrica, Vol. 19, 1951, pp. 250-272.

R. D. Driver, "A 'backwards' two-body problem of classical relativistic electrodynamics", Phys. Rev., Vol. 178, 1969, pp. 2051-2057.

R. D. Driver and M. J. Norris, "Note on uniqueness for a one-dimensional two-body problem of classical electrodynamics", Ann. Physics, Vol. 42, 1967, pp. 347-351.

S. P. Travis, "A one-dimensional two-body problem of classical electrodynamics", SIAM J. Appl. Math., Vol. 28, 1975, pp. 611-632.

R. D. Driver, "A two-body problem of classical electrodynamics: The one-dimensional case", Ann. Physics, Vol. 21, 1963, pp. 122-142.

D. K. Hsing, "An existence and uniqueness theorem for the one-dimensional backward two-body problem of electrodynamics", Phys. Rev., Vol. 16, No. 4, 1977, pp. 974-982.

4. Personnel

The research described in Part I was carried out by one of the principal investigators, Dr. A. Thowsen, and, in the case of certain stabilizability results [R-8], also by Mr. Ali Feliachi, a graduate research assistant in the School of Electrical Engineering at Georgia Institute of Technology. Mr. Feliachi received financial support from the NSF grant. He is now completing a Ph.D. dissertation in Electrical Engineering at Georgia Tech on the control of power system.

5. Publications

The following is a list of research papers written under NSF Grant ENG 78-12231:

R-1. A. Thowsen, "Control of time-delay systems", Proc. Eleventh Annual Southeastern Symposium on System Theory, Clemson University, SC, 1979, pp. 1-4.

R-2. A. Thowsen, "Characterization of state controllable time-delay systems with piecewise constant inputs. Part I: Derivation of general conditions", Int. J. Control, Vol. 31, No. 1, 1980, pp. 31-42.

R-3. A. Thowsen, "Characterization of state controllable time-delay systems with piecewise constant inputs. Part II: Analysis of second and third order systems", Int. J. Control, Vol. 31, No. 1, 1980, pp. 43-49.

R-4. A. Thowsen, "A result in the control of systems governed by linear differential-difference equations", Proc. Seventeenth Annual Allerton Conference on Communication, Control, and Computing, Urbana, IL, 1979.

R-5. A. Thowsen, "Zero state control of time-delay systems with piecewise constant inputs", submitted to IEEE Trans. Automatic Control.

R-6. A. Thowsen, "Computer control for low order time-delay systems", Int. J. Sys. Sci., Vol. 11, No. 8, 1980, pp. 1011-1019.

R-7. A. Thowsen, "Analysis and control of time-delay systems using finite dimensional approximations", to appear in Computers and Electrical Engineering, Vol. 8.

R-8. A. Feliachi and A. Thowsen, "Memoryless stabilization of linear delay-differential systems", to appear in IEEE Trans. Automatic Control, Vol. AC-26, No. 2, 1981.

CONTROL OF TIME-DELAY SYSTEMS

A. Thowsen

Georgia Institute of Technology, Atlanta, Georgia 30332

Some new conditions for function space null controllability of linear time-invariant delay systems with piecewise constant inputs will be presented. The results may find possible applications in digital industrial control.

Introduction

Many dynamic systems of engineering and scientific interest exhibit time delays. The delays may result from time-lags in the movement of mass and/or energy, delays in decision making, regeneration or signal transmission, or from delays introduced by the on-line computation required by some feedback control algorithms. The dynamic systems considered in this paper are those linear systems which can be adequately described by time-invariant differential-difference equations. Mathematically,

$$\dot{x}(t) = Ax(t) + Bx(t-1) + Cu(t), \quad t > 0 \tag{1}$$

$$y(t) = Hx(t) \tag{2}$$

where $x(t) \in R^n$ is the quantity to be regulated by the control function $u(\cdot)$ with $u(t) \in R^r$. $y(t) \in R^m$ is the measured system output and A,B,C, and H are constant matrices. The delay in (1) has been normalized to unity without loss of generality. Furthermore, linear systems with multiple commensurable time delays can also be modeled by the single delay system described in (1) [1]. Initial conditions for system (1) is the value of x on the interval $[-1,0]$, say $x(t) = \phi(t)$ where ϕ commonly is chosen as an element of the Banach space $C([-1,0], R^n)$ or the Hilbert space $M^2([-1,0],R^n)$. In those cases where $x(t)$ represents a deviation from desired operating conditions, a primary concern of the control theorist is the determination of mathematical conditions under which the deviation will remain small or become zero after some finite time. The linear quadratic optimal control problem for (1) has been solved by many researchers (e.g. see [2]) and necessary and sufficient conditions for bringing $x(t)$ or $y(t)$ to zero in finite time may have been first obtained by Zmood [3] although under the condition of pointwise degeneracy studied by Popov [4,5] $y(t)$ can in some cases be made identically equal to zero on a semi-infinite interval in the absence of any control input $u(\cdot)$. Extensions of Popov's ideas are found in articles by Asner and Halanay [6,7], Choudhury [8], Kappel [9] and Thowsen [10]. By enriching the class of feedback signals with augmented system outputs, it has also been shown how to obtain $x(t) \equiv 0$ for all $t \geq t_1$ (where t_1 = positive integer) for all initial conditions ϕ using delay-feedback control [11, 12]. These results all relate to systems with inputs restricted only to be continuous or square integrable time functions. In this paper we will study the situation which arises (e.g. in digital industrial control) when the system output or state is sampled uniformly and the control input to the continuous time process (1) is a piecewise constant vector function with constant value over each sampling period. This control problem is formulated in section 2; section 3 presents conditions for $x(t) \equiv 0$ on some interval $[t,\infty)$; and section 4 gives a complete solution for second order systems $(n = 2)$.

Problem Formulation

Consider the linear time-invariant delay system

$$\dot{x}(t) = Ax(t) + Bx(t-1) + Cu(t), \quad t > 0 \tag{3}$$

with $A \neq 0$, $B \neq 0$, $x(t) \in R^n$, and $u(t) \in R^r$. The initial conditions are $x(\tau) = \phi(\tau)$, $-1 \leq \tau \leq 0$ with $\phi \in C([-1,0];R^n)$ and the control function u is restricted by

$$u(t) = u(k-1) \text{ for } t \in (k-1,k], \ k=1,2,\ldots \tag{4}$$

to be a piecewise constant function. The dynamic behavior of (3) was first studied in [13]. In this paper we are concerned with the problem of controlling the state of the system, defined as the function $x_t = x(t+\theta)$, $-1 \leq \theta \leq 0$ at time t, to the zero state in the function space which serves as state space for the infinite dimensional system (3). This is commonly referred to as the problem of function space null controllability (f.s.n.c.). Let $x(t,\phi,u)$ denote the trajectory value in R^n at time t starting from initial condition ϕ with control u.

Definition

System (3) is said to be function space null controllable at time t if for each initial function $\phi \in C([-1,0],R^n)$ there exists an admissible control u (possibly dependent on ϕ) satisfying (4) such that the trajectory values $x(t,\phi,u) \equiv 0$ on $[t-1,\infty)$.

From previous work [13] it is known that the minimal time for system (3) to be function space null controllable is a positive integer. Furthermore, we have:

This work was supported by the National Science Foundation under Grant ENG78-12231.

Proposition 2.1

If system (3) is function space null controllable (f.s.n.c.) at time $\hat{t}$, then $\hat{t} \geq 3$.

Proof

Suppose $\hat{t} < 3$. Then (3) is also f.s.n.c. at $t = 2$ which implies that $x(t) = 0$ for $t \geq 1$. Since $Bx(t-1) = -Cu(1)$ for $t \in (1,2]$, evaluation at $t = 2$ yields $Bx(t) \equiv 0$ on $(0,1]$. By continuity $Bx(0) = 0$ for all $x(0) \in R^n$ which is impossible since $B \neq 0$.

The control problem for (3) can be formulated as a two-point boundary value problem which leads to a one-parameter family of integral equations as the necessary and sufficient conditions for function space null controllability. We use notation similar to that in [14]. Define for each positive integer p the matrices

$$A_p = \begin{bmatrix} A & & & 0 \\ B & A & & \\ & B & A & \\ 0 & & \ddots & \ddots \\ & & B & A \end{bmatrix}_{np \times np} \qquad B_p = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{np \times n}$$

$$J_p = \begin{bmatrix} 0 & & & 0 \\ I & 0 & & \\ & I & \ddots & \\ 0 & & \ddots & \ddots \\ & & I & 0 \end{bmatrix}_{np \times np} \qquad C_p = \begin{bmatrix} C & & 0 \\ & C & \\ 0 & & \ddots \\ & & & C \end{bmatrix}_{np \times nr}$$

$$Q_p = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}_{np \times n} \qquad U_p = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(p-1) \end{bmatrix}_{nr \times 1}$$

and set

$$y_i(t) \triangleq x(t+(i-1)), \quad 0 \leq t \leq 1, \quad i = 1,2,\ldots. \tag{5}$$

By continuity $y_{i+1}(0) = y_i(1)$. Define the np-dimensional vector $Y_p(t) = [y_1^T(t), y_2^T(t), \ldots, y_p^T(t)]^T$ and the n-dimensional vector $\Phi(t) = \phi(t-1)$ for all $t \in [0,1]$. System (3) can then be written as an ordinary differential equation.

$$\dot{Y}_p(t) = A_p Y_p(t) + B_p \Phi(t) + C_p U_p, \quad 0 \leq t \leq 1 \tag{6}$$

with boundary conditions

$$Y_p(0) = J_p Y_p(1) + \begin{bmatrix} \phi(0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7}$$

From (6) follows

$$Y_p(1) = e^{A_p(1-t)} Y_p(t) + \int_t^1 e^{A_p(1-\tau)} B_p \Phi(\tau)\,d\tau + \int_t^1 e^{A_p(1-\tau)}\,d\tau\, C_p U_p$$

$$Y_p(0) = e^{-A_p t} Y_p(t) - \int_0^t e^{-A_p \tau} B_p \Phi(\tau)\,d\tau - \int_0^t e^{-A_p \tau}\,d\tau\, C_p U_p$$

Upon substitution into (7), noting that J_p and A_p commute,

$$\begin{aligned}(I - J_p e^{A_p}) Y_p(t) &= \int_0^t e^{A_p(t-\tau)} B_p \Phi(\tau)\,d\tau \\ &+ \int_t^1 e^{A_p(t-\tau+1)} J_p B_p \Phi(\tau)\,d\tau + \int_0^t e^{A_p(t-\tau)}\,d\tau\, C_p U_p \\ &+ \int_t^1 e^{A_p(t-\tau+1)}\,d\tau\, J_p C_p U_p + e^{A_p t} \begin{bmatrix} \phi(0) \\ 0 \\ \vdots \\ 0 \end{bmatrix}\end{aligned} \tag{8}$$

The matrix $I - J_p e^{A_p}$ is diagonal with diagonal entries equal to 1. Consequently, $G = (I - J_p e^{A_p})^{-1}$ is a well defined matrix and it is easily shown that G commutes both with A_p and with J_p.

System (3) is function space null controllable at time $t = p$ (i.e. $x(t) \equiv 0$ for $t \geq p-1$) if and only if $Q_p^T Y_p(t) \equiv 0$. Consequently, a necessary and sufficient for f.s.n.c. at $t = p$ is that

$$\begin{aligned}Q_p^T \{ &\int_0^t G e^{A_p(t-\tau)} B_p \Phi(\tau)\,d\tau + \int_t^1 G e^{A_p(t-\tau+1)} J_p B_p \Phi(\tau)\,d\tau \\ &+ \int_0^t G e^{A_p(t-\tau)}\,d\tau\, C_p U_p + \int_t^1 G e^{A_p(t-\tau+1)}\,d\tau\, J_p C_p U_p \\ &+ e^{A_p t} \begin{bmatrix} \phi(0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \} \equiv 0 \quad \text{on } [0,1]\end{aligned} \tag{9}$$

Conditions for f.s.n.c.

In this section we present a set of simpler conditions on the system matrices which are both necessary and sufficient for the time-delay system to be function space null controllable at $t = p$.

Lemma

$$Q_p^T G - Q_p^T = Q_p^T G J_p e^{A_p}$$

Proof

$$G = (I - J_p e^{A_p})^{-1} = I + J_p (I - e^{A_p} J_p)^{-1} e^{A_p} = I + J_p G e^{A_p}$$

from the commutativity of J_p and G and the general matrix inversion identity [15].

Lemma

$Q_p^T A_p^i B_p = 0,\ i = 0,1,2,\ldots,n_p - 1$ is a necessary condition for f.s.n.c. at $t = p$.

Proof

Omitted for the sake of brevity.

With these lemmas we obtain the following general theorem:

Theorem

The time-delay system defined by (3) - (4) is f.s.n.c. at $t = p$ if and only if

$$Q_p^T A_p^i B_p = 0,\ i = 0,1,2,\ldots,n_p - 1$$

and there exists a control vector $U_p \in R^{nr}$ such that

$$Q_p^T G e^{A_p t}\{\int_0^1 e^{-A_p \tau}[B_p \phi(\tau) + C_p U_p]d\tau + \begin{bmatrix}\phi(0)\\ 0\\ \vdots\\ 0\end{bmatrix}\}$$

$$\equiv -Q_p^T e^{A_p t}\int_t^1 e^{-A_p \tau} d\tau C_p U_p \qquad \text{on } [0,1]$$

Remark

These conditions can be extended to the case of multiple delays by defining the quantities A_p, B_p and $\phi(t)$ as in [14].

An example of a system which for $p = 3$ satisfies these conditions was given in [13], namely

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t-1)\\ x_2(t-1)\end{bmatrix} + \begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

Second order systems (with $n = 2$) are of particular interest due to the following proposition.

Proposition 3.1

The scalar system (3) is never f.s.n.c.

Proof

If system is f.s.n.c. at some time t, then $bx(t) \equiv 0$ on $[t-2,t-1]$ which implies, since $b \neq 0$, that the system also is f.s.n.c. at t-1. The proposition follows directly.

Second Order Systems

In view of propositions 2.1 and 3.1, the simplest example of a function space null controllable system (3) - (4) is a second order system which is f.s.n.c. at $t = 3$. For this case a complete characterization is given by the following result.

Theorem

The second order time-delay system (3) - (4) is f.s.n.c. in minimal time (i.e. at $t = 3$) if and only if the matrices A and B and the vector $w = Cu$ satisfy

(1) $B^2 = 0$

(2) $AB = \gamma_1 B$, $\gamma_1 \in R$

(3) $BA = \gamma_2 B$, $\gamma_2 \in R$

(4) $w_i \in N(B)$ $\quad i,j \in \{1,2\}$, $i \neq j$
$w_j \in N^{\perp}(B)$

Proof

Omitted for the sake of brevity.

A canonical representation of such f.s.n.c. system is obtained in the coordinate system where $N(B) = \{x \in R^2,\ \dot{x}_2 = 0\}$ as

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12}\\ 0 & a_{22}\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1(t-1)\\ x_2(t-1)\end{pmatrix} + \begin{pmatrix}w_1\\ w_2\end{pmatrix}$$

where a_{11}, a_{12}, a_{22} and b are any real numbers and w_1, w_2 can take on any real values.

It may be possible for second order systems to first become f.s.n.c. for some $t \geq 4$. Those systems, however, must exhibit the same matrix structure in the A and B matrices as determined by conditions (1) - (3) in the last theorem.

Conclusion

Some new conditions for function space null controllability of linear time-invariant delay systems with piecewise constant inputs have been presented. The results may find possible applications in digital industrial control.

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Characterization of State Controllable Time-Delay Systems with Piecewise Constant Inputs. Part I: Derivation of General Conditions

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Abstract

The possibility of sampled-data feedback control for linear time-invariant multiple delay systems is studied using the constraint of piecewise constant control inputs. A split system representation is utilized to derive both necessary and sufficient conditions for control of the system state to the nullfunction in the state space.

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I. INTRODUCTION

Several dynamic systems of engineering interest are modeled by linear time-invariant differential-difference equations of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i x(t-i) + Cu(t) \tag{1.1}$$

where the delayed terms are caused by mass or energy transfer in the physical system or result from signal transmission delays.

Most control strategies for time-delay system (1.1) fall into three classes:

Class 1. Optimal linear quadratic control in which a cost functional quadratic in $x(t)$ and $u(t)$ is optimized. (See e.g. Alekal et al. [A-4].)

Class 2. Delay-feedback control in which past information about $x(t)$, assumed to be continuously available, is used to generate the control input. (Asner and Halanay [A-2,A-3], Popov [P-2], and Thowsen [T-2])

Class 3. Sampled-data feedback control with only sampled values of $x(t)$ available to the controller. This type of control algorithm, often associated with digital computer control of the system, leads to piecewise constant control inputs. (Thowsen and Perkins [T-1])

Of particular interest is a comparison of the storage (memory) requirements for the different control strategies. The state of system (1.1) at time t is $x_t = x(t+\theta)$, $\theta \in [-m,0]$. Class 1 controllers require generally that x_t is available to the controller at time t. For a physical process it is often desired to control x_t to the null state, i.e. to achieve $x_t = 0$ for some t. Class 2 controllers designed for this objective require generally that values of $x(t-i)$ is stored for a finite number of positive integers i. However, as time continuously progresses, these stored values must also be continuously updated. Class 3 controllers alleviate this problem as updating is required only at each sampling instant. From an implementational viewpoint class 3 controllers are therefore very attractive.

This paper determines the class of time-delay systems for which the state x_t can be controlled to the null state in finite time using piecewise constant system inputs (as would result from sampled-data feedback control). In particular, in section II system (1.1) is represented by a "split system" consisting of a high order finite dimensional system and an associated two point boundary condition. Section III considers conditions for controlling the state to zero. In that section the standard concept of function space null controllability given below is employed.

Definition

System (1.1) is function space null controllable if for any given initial function ϕ defined on $[-m,0]$ there exist a finite time t_1 and an admissible control $u(\cdot)$ defined on $(0,t_1]$ such that $x(t_1,\phi,u) = 0$ on $[t_1-m,t_1]$ where $x(t,\phi,u)$ is the solution to (1.1) at time t with initial condition ϕ and control u.

Finally, in section IV we obtain simpler necessary and sufficient conditions for controllability and an algebraic necessity condition. The implications for the system structure of function space null controllable systems with piecewise constant inputs are studied in a companion article.

II. THE "SPLIT SYSTEM"

The unique solution to the system of linear time-invariant differential-difference equation

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i x(t-i) + Cu(t) \ , \quad t > 0$$

$$x(t) = \phi(t) \ , \quad t \in [-m,0] \tag{2.1}$$

$$\phi \in C([-m,0];R^n) \ , \quad B_m \neq 0$$

is also the unique solution to a certain higher order system of ordinary differential equation with an associated two point boundary value condition. This new system of equations was called the "split system" in a paper by Charrier and Haugazeau [C-1] but has also appeared in earlier work by Popov [P-1], Zmood [Z-1], and Asner and Halanay [A-1]. The split system equations are here developed for the multiple delay system (2.1) with piecewise constant input functions. Let $x(t) \in R^n$ and $u(t) \in R^r$ and define

$$u(t+i) \triangleq u_i = \text{constant} \ , \quad 0 < t \leq 1 \ , \quad i = 0,1,2,\dots. \tag{2.2}$$

Furthermore, define

$$\Phi_i(t) \triangleq \phi(t-i) \ , \quad 0 \leq t \leq 1 \ , \quad i = 1,2,\dots,m,$$

and

$$y_i(t) \triangleq x(t+(i-1)) \ , \quad 0 < t \le 1 \ , \quad i = 1,2,3,\ldots.$$

By continuity of $x(\cdot)$,

$$y_{i+1}(0) = y_i(1) \ , \quad i = 1,2,3,\ldots. \tag{2.3}$$

The solution $x(\cdot)$ to the multiple delay system (2.1) can be determined by solving successively for $y_i(t)$, $i = 1,2,\ldots,$ on the interval $[0,1]$ from the following set of equations:

$$\dot{y}_1(t) = Ay_1(t) + \sum_{i=1}^{m} B_i\Phi_i(t) + Cu_0$$

$$\dot{y}_2(t) = Ay_2(t) + B_1y_1(t) + \sum_{i=2}^{m} B_i\Phi_{i-1}(t) + Cu_1$$

$$\dot{y}_3(t) = Ay_3(t) + B_1y_2(t) + B_2y_1(t) + \sum_{i=3}^{m} B_i\Phi_{i-2}(t) + Cu_2$$

$$\cdots\cdots$$

$$\dot{y}_k(t) = Ay_k(t) + B_1y_{k-1}(t) + B_2y_{k-2}(t) + \ldots + B_{k-1}y_1(t) + \sum_{i=k}^{m} B_i\Phi_{i-k+1}(t) + Cu_{k-1} \qquad \text{for } 1 \le k \le m \tag{2.4}$$

$$\dot{y}_{m+1}(t) = Ay_{m+1}(t) + B_1y_m(t) + B_2y_{m-1}(t) + \ldots + B_my_1(t) + Cu_m$$

$$\cdots\cdots$$

$$\dot{y}_p(t) = Ay_p(t) + B_1y_{p-1}(t) + B_2y_{p-2}(t) + \ldots + B_my_{p-m}(t) + Cu_{p-1}$$

$$\cdots\cdots \qquad \text{for } p \ge m+1.$$

Equ. (2.3)-(2.4) constitute the split system for system (2.1). For every integer $p \geq 1$ define the following vectors and matrices (shown here for $p > m+1$):

$$
A_p \triangleq \begin{bmatrix}
A & & & & & & & & & & & \\
B_1 & A & & & & & & \bigcirc & & & & \\
B_2 & B_1 & A & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \\
B_m & B_{m-1} & \cdot & \cdot & \cdot & B_1 & A & & & & & \\
0 & B_m & B_{m-1} & \cdot & \cdot & \cdot & B_1 & A & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
0 & \cdot & \cdot & \cdot & B_m & B_{m-1} & \cdot & \cdot & \cdot & \cdot & B_1 & A
\end{bmatrix}_{np \times np}
$$

$$
B_p \triangleq \begin{bmatrix}
B_1 & B_2 & B_3 & \cdot & \cdot & \cdot & B_m \\
B_2 & B_3 & \cdot & \cdot & \cdot & B_m & 0 \\
B_3 & \cdot & \cdot & \cdot & B_m & 0 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & & \bigcirc & \\
\cdot & \cdot & \cdot & & & & \\
B_m & 0 & & & & & \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & & & \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0
\end{bmatrix}_{np \times nm}
$$

$$\psi(t) \triangleq \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \phi_m(t) \end{bmatrix}_{nm \times 1}, \qquad U_p \triangleq \begin{bmatrix} u_0 \\ u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{p-1} \end{bmatrix}_{pr \times 1}$$

$$C_p \triangleq \begin{bmatrix} C & & & \\ & C & \bigcirc & \\ & & \ddots & \\ & \bigcirc & & C \end{bmatrix}_{np \times pr}$$

$$J_p \triangleq \begin{bmatrix} 0 & & & & \\ I_n & \cdot & & \bigcirc & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & \bigcirc & & \cdot & \cdot \\ & & & I_n & 0 \end{bmatrix}_{np \times np}$$

and

$$Y_p(t) \triangleq \begin{bmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_p(t) \end{bmatrix}_{np \times 1}$$

With this notation it is easily established that the split system corresponding to equ. (2.1) consists of the np ordinary differential equations

$$\dot{Y}_p(t) = A_p Y_p(t) + B_p \psi(t) + C_p U_p \,, \quad 0 < t \leq 1 \tag{2.5}$$

and the two point boundary condition

$$Y_p(0) = J_p Y_p(1) + \begin{bmatrix} \phi(0) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \tag{2.6}$$

III. CONDITIONS FOR FUNCTION SPACE NULL CONTROLLABILITY

The split system can be used to derive necessary and sufficient conditions for function space null controllability. Equ. (2.5) has the solution

$$Y_p(t) = e^{A_p t} Y_p(0) + \int_0^t e^{A_p(t-\tau)} B_p \psi(\tau) d\tau + \int_0^t e^{A_p(t-\tau)} d\tau C_p U_p \quad .$$

So upon substitution for $Y_p(0)$ from (2.6)

$$Y_p(t) = e^{A_p t} J_p Y_p(1) + e^{A_p t}\psi_0 + \int_0^t e^{A_p(t-\tau)} B_p \psi(\tau) d\tau$$
$$+ \int_0^t e^{A_p(t-\tau)} d\tau C_p U_p \tag{3.1}$$

where

$$\psi_0 \triangleq \begin{bmatrix} \phi(0) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} .$$

Define the $np \times n$ vector Q by $Q_p^T \triangleq [0\ldots0\ \overbrace{I\ldots\ldots I}^{m \text{ times}}]$. Then $x(t) \equiv 0$ on $p-m \le t \le p$ is equivalent to $Q_p^T Y_p(t) \equiv 0$ on $[0,1]$. In [T-1] Thowsen and Perkins proved the following lemma.

Lemma 3.1

If the single delay system

$$\dot{x}(t) = Ax(t) + Bx(t-1) + Cu(t)$$

$$x(t) = \phi(t) \ , \quad t \in [-1,0]$$

with

$$u(t) = u_k \ \forall \ t \in (k,k+1]$$

is function space null controllable at time t_1 but not at any $t < t_1$, then t_1 is a non-negative integer.

A similar result holds for multiple delay systems.

Lemma 3.2

If system (2.1) with piecewise constant control

$$u(t) = u_k \ \forall \ t \in (k,k+1]$$

is function space null controllable at time t_1 but not at any $t < t_1$, then t_1 is a positive integer $\geq m+1$.

Proof

Define the nm dimensional vector

$$z(t) \triangleq \begin{bmatrix} x(t) \\ x(t-1) \\ x(t-2) \\ \cdot \\ \cdot \\ \cdot \\ x(t-m+1) \end{bmatrix}$$

and rewrite (2.1) as the single delay system

$$\dot{z}(t) = \begin{bmatrix} A & B_1 & B_2 & \cdots & B_{m-1} \\ & A & B_1 & \cdots & B_{m-2} \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & B_1 \\ & & & & A \end{bmatrix} z(t)$$

$$+ \begin{bmatrix} B_m & & & & \\ B_{m-1} & B_m & & 0 & \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & \\ B_1 & B_2 & \cdots & \cdots & B_m \end{bmatrix} z(t-1)$$

$$+ \begin{bmatrix} C & & & \\ & C & & 0 \\ & & \ddots & \\ & 0 & & C \end{bmatrix} \begin{bmatrix} u(t) \\ u(t-1) \\ \vdots \\ u(t-m+1) \end{bmatrix} \tag{3.2}$$

System (3.2) is function space null controllable at t_1 if and only if $z(t) \equiv 0$ on $[t_1-1,t_1]$ which is equivalent to $x(t) \equiv 0$ on $[t_1-m,t_1]$. By lemma 3.1 the maximal time set where $z_t = 0$ is an interval of the form $[k,\infty)$ where k is an integer. From the definition of t_1, $t_1 = k$. Finally, since $B_m \neq 0$

and since $x_{t_1} = 0$ must hold for all initial conditions, $t_1 \geq m+1$.

From (2.5)

$$Y_p(1) = e^{A_p(1-t)}Y_p(t) + \int_t^1 e^{A_p(1-\tau)}B_p\psi(\tau)d\tau$$

$$+ \int_t^1 e^{A_p(1-\tau)}C_pU_p d\tau$$

and upon substitution into (3.1)

$$Y_p(t) - e^{A_pt}J_pe^{A_p(1-t)}Y_p(t) = e^{A_pt}J_p \int_t^1 e^{A_p(1-\tau)}B_p\psi(\tau)d\tau$$

$$+ e^{A_pt}J_p \int_t^1 e^{A_p(1-\tau)}C_pU_p d\tau + e^{A_pt}\psi_0$$

$$+ \int_0^t e^{A_p(t-\tau)}B_p\psi(\tau)d\tau + \int_0^t e^{A_p(t-\tau)}d\tau C_pU_p \tag{3.3}$$

The following three lemmas are useful in simplifying (3.3).

Lemma 3.3

J_p and e^{A_pt} commute.

Proof

Multiplication shows that $J_pA_p = A_pJ_p$. But then $J_pA_p^{\,2} = A_pJ_pA_p = A_p^{\,2}J_p$ and in general $J_pA_p^{\,i} = A_p^{\,i}J_p$, $i = 0,1,2,\ldots.$ Hence $J_pe^{A_pt} = e^{A_pt}J_p$.

The matrix $I - J_pe^{A_p}$ is a lower triangular matrix with unit elements along the main diagonal. Hence its inverse

$$G_p \triangleq (I - J_pe^{A_p})^{-1}$$

exists.

Lemma 3.4

G_p and J_p commute.

Proof

$G_pJ_p = (I - J_pe^{A_p})^{-1}J_p = J_p(I - J_pe^{A_p})^{-1} = J_pG_p$ since J_p commutes with $I - J_pe^{A_p}$ by Lemma 3.3 and hence also with $(I - J_pe^{A_p})^{-1}$.

Lemma 3.5

G_p and A_p commute.

Proof

$G_pA_p = (I - J_pe^{A_p})^{-1}A_p = A_p(I - J_pe^{A_p})^{-1} = A_pG_p$ since A_p commutes with $I - J_pe^{A_p}$ by Lemma 3.3 and hence also with $(I - J_pe^{A_p})^{-1}$.

Equ. (3.3) can now be rewritten in more compact form as

$$\begin{aligned} Y_p(t) = & \int_0^t G_pe^{A_p(t-\tau)}B_p\psi(\tau)d\tau \\ & + \int_t^1 G_pJ_pe^{A_p}e^{A_p(t-\tau)}B_p\psi(\tau)d\tau \\ & + \int_0^t G_pe^{A_p(t-\tau)}C_pU_pd\tau + \int_t^1 G_pJ_pe^{A_p}e^{A_p(t-\tau)}C_pU_pd\tau \\ & + G_pe^{A_pt}\psi_0 \end{aligned} \tag{3.4}$$

From the general matrix inversion identity (F-1)

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}$$

follows that

$$G_p = (I - J_p e^{A_p})^{-1} = I + J_p(I - e^{A_p}J_p)^{-1}e^{A_p}$$

$$= I + G_p J_p e^{A_p} \tag{3.5}$$

by the commutativity of J_p and e^{A_p} and of G_p and J_p established in lemmas 3.3 and 3.4.

Substituting for $J_p G e^{A_p}$ from (3.5) into (3.4) gives

$$Y_p(t) = \int_0^t G_p e^{A_p(t-\tau)} B_p \psi(\tau) d\tau$$

$$+ \int_t^1 G_p e^{A_p(t-\tau)} B_p \psi(\tau) d\tau - \int_t^1 e^{A_p(t-\tau)} B_p \psi(\tau) d\tau$$

$$+ \int_0^t G_p e^{A_p(t-\tau)} C_p U_p d\tau + \int_t^1 G_p e^{A_p(t-\tau)} C_p U_p d\tau$$

$$- \int_t^1 e^{A_p(t-\tau)} C_p U_p d\tau + G_p e^{A_p t} \psi_0$$

$$= \int_0^1 G_p e^{A_p(t-\tau)} B_p \psi(\tau) d\tau - \int_t^1 e^{A_p(t-\tau)} B_p \psi(\tau) d\tau$$

$$+ \int_0^1 G_p e^{A_p(t-\tau)} C_p U_p d\tau - \int_t^1 e^{A_p(t-\tau)} C_p U_p d\tau$$

$$+ G_p e^{A_p t} \psi_0$$

From lemma 3.2 follows that system (2.1) is function space null controllable if and only if there exists a positive integer $p \geq m+1$ such that $x_p = 0$ (i.e. $x(t) \equiv 0$ on $[p-m,p]$). Since this condition is identical to $Q_p^T Y_p(t) \equiv 0$, we have:

Theorem 3.1

System (2.1) with the piecewise constant controls $u(t) = u_k \; \forall \; t \in (k,k+1]$ is function space null controllable if and only if there exists a finite positive integer $p > m$ such that

$$\int_0^1 Q_p^T G_p e^{A_p(t-\tau)} B_p \psi(\tau) d\tau - \int_t^1 Q_p^T e^{A_p(t-\tau)} B_p \psi(\tau) d\tau$$

$$+ \int_0^1 Q_p^T G_p e^{A_p(t-\tau)} C_p U_p d\tau - \int_t^1 Q_p^T e^{A_p(t-\tau)} C_p U_p d\tau$$

$$+ Q_p^T G_p e^{A_p t} \psi_0 \equiv 0 \quad \text{on } [0,1].$$

Corollary 3.1

If the condition in theorem 3.1 is satisfied for some positive integer $p > m$, then the condition is also satisfied for all integers greater than p.

IV. AN ALGEBRAIC CONDITION FOR FUNCTION SPACE NULL CONTROLLABILITY

In this section it is shown that the integral criterion in theorem 3.1 requires an algebraic condition to be satisfied. This condition is then an algebraic necessity condition for function space null controllability and, indirectly, for closed-loop sampled-data control. By theorem 3.1 and lemma 3.5 the condition

$$Q_p^T e^{A_p t} \{ \int_0^1 G_p e^{-A_p \tau} B_p \psi(\tau) d\tau - \int_t^1 e^{-A_p \tau} B_p \psi(\tau) d\tau$$

$$+ \int_0^1 G_p e^{-A_p \tau} d\tau C_p U_p - \int_t^1 e^{-A_p \tau} d\tau C_p U_p + G_p \psi_0 \} \equiv 0 \qquad (4.1)$$

on [0,1] is both necessary and sufficient for system (2.1) with piecewise constant controls given by (2.2) to be function space null controllable at time p. Take as initial condition $\phi \in C([-m,0];R^n)$ any continuous function such that for some $\varepsilon > 0$ and $\hat{t} \in (0,1)$

$$\phi(t) = \begin{cases} 0 & t \in [-1-k, -\hat{t}-\varepsilon-k] \\ q & t = -\hat{t}-k \\ 0 & t \in [-\hat{t}+\varepsilon-k, -k] \end{cases}$$

for all $k = 0,1,2...,m-1$; with $q \in R^n$. For notational simplicity define

$$f(t) \triangleq \int_0^1 G_p e^{-A_p \tau} B_p \psi(\tau) d\tau - \int_t^1 e^{-A_p \tau} B_p \psi(\tau) d\tau$$

$$+ \int_0^1 G_p e^{-A_p \tau} d\tau C_p U_p - \int_t^1 e^{-A_p \tau} d\tau C_p U_p$$

Differentiation of (4.1) yields

$$Q_p^T A_p e^{A_p t} f(t) + Q_p^T B_p \psi(t) + Q_p^T C_p U_p \equiv 0 \quad \text{on } [0,1]. \tag{4.2}$$

Evaluation of (4.2) for $t \in [0,\hat{t}-\varepsilon]$ gives

$$Q_p^T A_p e^{A_p t} \Big\{ \int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} G_p e^{-A_p \tau} B_p \psi(\tau) d\tau - \int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} e^{-A_p \tau} B_p \psi(\tau) d\tau$$

$$+ \int_0^1 G_p e^{-A_p \tau} d\tau C_p U_p - \int_t^1 e^{-A_p \tau} d\tau C_p U_p \Big\} + Q_p^T C_p U_p$$

$$= Q_p^T A_p e^{A_p t} \int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} (G_p - I) e^{-A_p \tau} B_p \psi(\tau) d\tau$$

$$+ Q_p^T e^{A_p t} \left\{ G_p (I - e^{-A_p}) + (e^{-A_p} - e^{-A_p t}) \right\} C_p U_p + Q_p^T C_p U_p$$

$$= Q_p^T A_p e^{A_p t} \int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} (G_p - I) e^{-A_p \tau} B_p \psi(\tau) d\tau$$

$$+ Q_p^T e^{A_p t} \left\{ G_p (I - e^{-A_p}) + e^{-A_p} \right\} C_p U_p = 0.$$

There exists an $M > 0$ such that

$$\Big\| Q_p^T A_p e^{A_p t} \int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} (G_p - I) e^{-A_p \tau} B_p \psi(\tau) d\tau \Big\| < \varepsilon M \tag{4.3}$$

for all $t \in [0,1]$. Letting $\varepsilon \to 0$ implies

$$Q_p^T e^{A_p t} \{G_p(I - e^{-A_p}) + e^{-A_p}\} C_p U_p \equiv 0 \quad \text{on } [0,\hat{t}-\varepsilon]$$

Since this function is analytic on [0,1] and identically zero on a closed sub-interval, it must be identically zero on [0,1], i.e.

$$Q_p^T e^{A_p t} \{G_p(I - e^{-A_p}) + e^{-A_p}\} C_p U_p \equiv 0 \quad \text{on } [0,1]$$

Evaluation of (4.2) at $t = \hat{t}$ then gives

$$Q_p^T A_p e^{A_p t} \{\int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} (G_p - I)e^{-A_p \tau} B_p \psi(\tau)d\tau$$

$$+ Q_p^T e^{A_p \hat{t}} \{G_p(I - e^{-A_p}) + e^{-A_p}\} C_p U_p$$

$$- Q_p^T B_p q$$

$$= Q_p^T A_p e^{A_p \hat{t}} \{\int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} (G_p - I)e^{-A_p \tau} B_p \psi(\tau)d\tau$$

$$+ Q_p^T B_p q \equiv 0$$

Equ. (4.3) then implies that

$$Q_p^T B_p q = 0$$

and since $q \in R^n$ is arbitrary

$$Q_p^T B_p = 0 \tag{4.4}$$

Next, differentiate (4.1) twice with respect to time, using (4.4) to write the result as

$$Q_p^T A_p^2 e^{A_p t} f(t) + Q_p^T A_p B_p \psi(t) + Q_p^T A_p C_p U_p = 0 \tag{4.5}$$

Evaluation of (4.5) at $t \in [0,\hat{t}-\varepsilon]$ gives

$$Q_p^T A_p^2 e^{A_p t} \left\{ \int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} (G_p - I) e^{-A_p \tau} B_p \psi(\tau) d\tau \right\}$$

$$+ Q_p^T A_p^2 e^{A_p t} \int_0^1 G_p e^{-A_p \tau} d\tau C_p U_p - Q_p^T A_p^2 e^{A_p t} \int_t^1 e^{-A_p \tau} d\tau C_p U_p$$

$$+ Q_p^T A_p C_p U_p$$

$$= Q_p^T A_p^2 e^{A_p t} \left\{ \int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} (G_p - I) e^{-A_p \tau} B_p \psi(\tau) d\tau \right\}$$

$$+ Q_p^T A_p e^{A_p t} \left\{ G_p (I - e^{-A_p}) + (e^{-A_p} - e^{-A_p t}) \right\} C_p U_p$$

$$+ Q_p^T A_p C_p U_p$$

$$= Q_p^T A_p^2 e^{A_p t} \left\{ \int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} (G_p - I) e^{-A_p \tau} B_p \psi(\tau) d\tau \right\}$$

$$+ Q_p^T A_p e^{A_p t} [G(I - e^{-A_p}) + e^{-A_p}] C_p U_p = C$$

Since the first term on the left side of the last equality is bounded for any $t \in [0,1]$ by $N\varepsilon$ for some $N > 0$, it follows by letting $\varepsilon \to 0$ that

$$Q_p^T A_p e^{A_p t} [G_p (I - e^{-A_p}) + e^{-A_p}] C_p U_p \equiv 0 \quad \text{on } [0,\hat{t}-\varepsilon]$$

and by analyticity on the whole interval $[0,1]$. Evaluating (4.5) at $t=\hat{t}$ then gives

$$Q_p^T A_p^2 e^{A_p \hat{t}} \left\{ \int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} (G_p - I) e^{-A_p \tau} B_p \psi(\tau) d\tau \right\}$$

$$- Q_p^T A_p B_p \psi(\hat{t}) = 0$$

Let $\varepsilon \to 0$ to obtain

$$Q_p^T A_p B_p \psi(\hat{t}) = Q_p^T A_p B_p q = 0$$

Since $q \in R^n$ is arbitrary,

$$Q_p^T A_p B_p = 0. \tag{4.6}$$

In general, differentiating (4.1) k times with respect to time t and using $Q_p^T A_p^{k-2} B_p = 0$ yield

$$Q_p^T A_p^k e^{A_p t} f(t) + Q_p^T A_p^{k-1} B_p \psi(t) + Q_p^T A_p^{k-1} C_p U_p = 0 \tag{4.7}$$

Evaluating at $t \in [0, \hat{t}-\varepsilon]$, letting $\varepsilon \to 0$ and invoking analyticity imply that

$$Q_p^T A_p^{k-1} [G_p (I - e^{-A_p}) + e^{-A_p}] C_p U_p \equiv 0 \quad \text{on } [0,1]$$

Finally, evaluating (4.7) at $t = \hat{t}$ and letting $\varepsilon \to 0$ result in the relation

$$Q_p^T A_p^{k-1} B_p \psi(\hat{t}) = Q_p^T A_p^{k-1} B_p q = 0 \quad \text{for all } q \in R^n.$$

Hence,

$$Q_p^T A_p^{k-1} B_p = 0$$

By the Cayley-Hamilton theorem there exist at most np such linearly independent conditions. These conditions constitute algebraic necessity conditions for function space null controllability.

Theorem 4.1

Necessary conditions for system (2.1) with piecewise constant controls (2.2) to be function space null controllable at time $t = p > 0$ (p = integer) are

$$Q_p^T A_p^i B_p = 0 , \quad i = 0,1,\ldots,np-1.$$

Theorem 4.1 will be employed in the next section to determine the structural properties of time-delay systems which can be controlled using piecewise constant inputs. Furthermore, the conditions of theorem 4.1 lead to an improved statement of the necessary and sufficient conditions for function space null controllability.

Theorem 4.2

System (2.1)-(2.2) is function space null controllable if and only if there exist a positive integer p and a control vector $U_p \in R^{pr}$ such that

$$(1) \quad Q_p^T A_p^i B_p = 0 , \quad i = 0,1,2,\ldots,np-1$$

and

$$(2) \quad Q_p^T G_p e^{A_p t} \left\{ \int_0^1 e^{-A_p \tau} B_p \psi(\tau) d\tau + \int_0^1 e^{-A_p \tau} d\tau C_p U_p + \psi_0 \right\}$$

$$= Q_p^T e^{A_p (t-\tau)} d\tau C_p U_p \;\; \forall \; t \in [0,1].$$

Proof

From theorem 4.1 obtain $Q_p^T e^{A_p t} B_p \equiv 0$ and substitute into the integral condition in theorem 3.1.

Summary

The control of linear multiple-delay systems to the null state was studied under the constraint of piecewise constant control signals. This type of constraint arises e.g. in sampled-data feedback control of time-delay systems. It was shown how the infinite dimensional system model can be represented by a split system consisting of a finite dimensional system together with a two point boundary condition. Based on this system representation both necessary and sufficient conditions for controlling the state to the null function in the state space were derived.

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Characterization of State Controllable Time-Delay Systems with Piecewise Constant Inputs. Part II: Analysis of Second and Third Order Systems

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Abstract

An algebraic condition necessary for controlling the state of linear time-invariant delay systems to the nullfunction is used to determine the concomitant structural properties of second and third order systems. It is shown that the characteristic quasipolynomials for the undriven systems are polynomials of second and third order, respectively.

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I. INTRODUCTION

In this paper the structural properties of linear time-invariant delay systems of the form

$$\dot{x}(t) = Ax(t) + Bx(t-1) + Cu(t), \quad t > 0 \tag{1.1}$$

$$u(t) = u_k \quad t \in (k,k+1]$$

$$x(t) = \phi(t) \quad t \in [-1,0]$$

$$x(t) \in R^n, \quad u(t) \in R^r$$

which are controllable to the zero state will be studied. Since any multiple delay system described by

$$\dot{x}(t) = \sum_{i=0}^{m} B_i x(t-ih) + Cu(t)$$

can also be modeled as a single delay system [R-1], attention is restricted to system (1.1). Furthermore, for computational simplicity, only second and third order systems will be studied. In a companion paper (T-1) it was shown that the state of (1.1) was controllable to the zero state at some finite time (i.e. the system was function space null controllable) only if there exists a positive integer p such that

$$Q_p^T A_p^i B_p = 0 \ , \quad i = 0,1,2,\ldots,np-1 \tag{1.2}$$

with

$$Q_p^T = [0. \ . \ .0 \ I]_{n \times np}$$

$$A_p = \begin{bmatrix} A & & & & \\ B & A & & \bigcirc & \\ & B & \cdot & & \\ & & \cdot & \cdot & \\ & \bigcirc & & \cdot & \cdot \\ & & & & \cdot \quad \cdot \\ & & & & B \quad A \end{bmatrix}_{np \times np}$$

and

$$B_p = \begin{bmatrix} B_p \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}_{np \times n}.$$

In this paper canonical representations for the time-delay systems which satisfy (1.2) will be presented and a complete characterization of their characteristic quasi polynomials will be derived. Section II gives the results for second order systems while section III contains the corresponding results for third order time-delay systems.

II. STRUCTURAL CONDITIONS FOR SECOND ORDER SYSTEMS

In this section second order systems (for which $n = 2$) will be studied to determine the structural requirements imposed upon the system matrices by the algebraic necessity condition (1.2).

For $k \geq p-1$ let $(A_p{}^k)_*$ denote the $n \times n$ matrix in matrix block position $(p,1)$ of the $np \times np$ matrix $A_p{}^k$, i.e.

$$A_p{}^k = \begin{bmatrix} A^k & \cdot & & \bigcirc & \\ \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ (A_p{}^k)_* & \cdot & \cdot & \cdot & A^k \end{bmatrix}$$

$(A_p{}^k)_* = S_k(A,B)$ where $S_k(A,B)$ denotes the sum of all possible products formed by the matrices A and B by making each product consist of B taken $p-1$ times and A taken $k-p+1$ times. Altogether the sum $S_k(A,B)$ contains $\binom{k}{p-1}$ terms. Since

$$Q_p{}^T A_p{}^k B_p = \begin{cases} 0 & k = 0,1,2,\ldots,p-2 \\ (A_p{}^k)_* B & k = p-1, p-2, \ldots \end{cases}$$

$Q_p{}^T A_p{}^{p-1} B_p = S_{p-1}(A,B)B = B^p = 0$ by condition (1.2). The matrix B is of dimension 2×2 and hence $B^2 = 0$. Since B is nilpotent with index 2,

$$Q_p{}^T A_p{}^{2(p-1)} B_p = S_{2p-2}(A,B)B = \underbrace{BABA\ldots BAB}_{\text{BA appears } p-1 \text{ times}} = 0 \tag{2.1}$$

by condition (1.2). However, $(BA)^2 = \gamma BA$ for some $\gamma \in R$ because B is singular and condition (2.1) becomes

$$\gamma^{p-2} BAB = 0 \tag{2.2}$$

Since B is nilpotent with index 2, it has a canonical representation

$$B = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \quad b \neq 0 \text{ in some coordinate}$$

system. With

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

$$(BA)^2 = \begin{bmatrix} (ba_{21})^2 & b^2 a_{21} a_{22} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad BAB = \begin{bmatrix} 0 & b^2 a_{21} \\ 0 & 0 \end{bmatrix}$$

so if $\gamma = 0$, then $(BA)^2 = 0$ implies $BAB = 0$. Consequently, (2.2) always implies that $BAB = 0$.

The characteristic quasipolynomial $\phi(\lambda, e^{-\lambda})$ for system (1.1) is invariant under coordinate transformations (similarity transformations) and can therefore be obtained directly from the canonical representation of the system.

$$\phi(\lambda, e^{-\lambda}) = \det(\lambda I - A - Be^{-\lambda}) = \begin{vmatrix} \lambda - a_{11} & \lambda - a_{12} - be^{-\lambda} \\ 0 & \lambda - a_{22} \end{vmatrix}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22}$$

Hence, the characteristic quasipolynomial reduces to a polynomial for those second order time-delay systems which are function space null controllable with piecewise constant controls.

III. STRUCTURAL CONDITIONS FOR THIRD ORDER SYSTEMS

In this section function space null controllable single-delay systems of the form

$$\dot{x}(t) = Ax(t) + Bx(t-1) + Cu(t),\ B \neq 0,\ t \in 0 \tag{3.1}$$

$$x(t) = \phi(t) \quad t \in [-1,0]$$

$$u(t) = u_k \quad t \in (k,k+1],\ k = 0,1,2,\ldots \tag{3.2}$$

with $x(t) \in R^3$ are studied.

Condition (1.2) determines that B is singular since $Q_p{}^T A_p{}^{p-1} B_p = S_{p-1}(A,B)B = B^p = 0$. Depending on the dimension of $N(\mathcal{B})$, the nullspace of linear operator $\mathcal{B}$ represented by matrix B, two cases must be considered.

Case 1 $\quad \dim N(\mathcal{B}) = 2$

In a particular coordinate system B has the representation

$$B = \begin{bmatrix} 0 & 0 & b_1 \\ 0 & 0 & b_2 \\ 0 & 0 & b_3 \end{bmatrix} \text{ with } (b_1,b_2,b_3) \neq (0,0,0)$$

Since $B^p = b_3^{p-1}B = 0$, $b_3 = 0$. Then

$$\mathcal{B}^2 = 0 \tag{3.3}$$

From (1.2)

$$Q_p{}^T A_p{}^{2p-2} B_p = \underbrace{BABA\ldots BAB}_{\text{BA appears } p-1 \text{ times}} = 0 \tag{3.4}$$

With

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } \Delta = b_1 a_{31} + b_2 a_{32} \tag{3.4}$$

(3.4) becomes $Q_p^T A_p^{2p-2} B_p = \Delta^{p-1} B = 0$ which means that $\Delta = 0$. Then

$$BAB = \begin{bmatrix} 0 & 0 & b_1\Delta \\ 0 & 0 & b_2\Delta \\ 0 & 0 & 0 \end{bmatrix} = 0 \tag{3.5}$$

Finally, in view of (3.3) and (3.5)

$$S_{3p-3}(A,B)B = \underbrace{BA^2BA^2\ldots BA^2B}_{BA^2 \text{ appears p-1 times}}$$

and by condition (1.2)

$$Q_p^T A_p^{3p-3} B_p = BA^2BA^2\ldots BA^2B = 0 \tag{3.6}$$

Since

$$BA^2B = \begin{bmatrix} 0 & 0 & b_1\delta \\ 0 & 0 & b_2\delta \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$BA^2 = \begin{bmatrix} b_1c_1 & b_1c_2 & b_1c_3 \\ b_2c_1 & b_2c_2 & b_2c_3 \\ 0 & 0 & 0 \end{bmatrix}$$

with

$$c_1 = a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31}$$

$$c_2 = a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32}$$

$$c_3 = a_{31}a_{13} + a_{32}a_{23} + a_{33}^2$$

$$\delta = b_1c_1 + b_2c_2,$$

(3.6) gives $\delta^{p-1}B = 0$ so

$$\delta = 0 \tag{3.7}$$

The characteristic quasipolynomial $\phi(\lambda,e^{-\lambda})$ for a third order system of form (3.1) which is controllable to the null state using (3.2) is then

$$\phi(\lambda,e^{-\lambda}) = \det(\lambda I - A - Be^{-\lambda}) = \begin{bmatrix} \lambda-a_{11} & -a_{12} & -a_{13}-b_1e^{-\lambda} \\ -a_{21} & \lambda-a_{22} & -a_{23}-b_2e^{-\lambda} \\ -a_{31} & -a_{32} & \lambda-a_{33} \end{bmatrix}$$

$$= \text{polynomial in } \lambda - (a_{31}b_1 + a_{32}b_2)\lambda e^{-\lambda}$$

$$+ (a_{11}a_{32}b_2 + a_{22}a_{31}b_1 - a_{12}a_{31}b_2 - a_{21}a_{32}b_1)e^{-\lambda}$$

$$= \text{polynomial in } \lambda - (a_{11}a_{31}b_1 + a_{22}a_{32}b_2 + a_{12}a_{31}b_2 + a_{21}a_{32}b_1)e^{-\lambda}$$

since $\Delta = b_1a_{31} + b_2a_{32} = 0$. Adding $(b_1a_{31} + b_2a_{32})a_{33} = 0$ to $\phi(\lambda,c^{-\lambda})$ we obtain

$$\phi(\lambda,e^{-\lambda}) = \text{polynomial in } \lambda - (a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33})b_1e^{-\lambda}$$

$$- (a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})b_2e^{-\lambda}$$

$= \text{polynomial in } \lambda$ by (3.7)

Hence, the characteristic quasipolynomial reduces to a polynomial in λ.

<u>Case 2</u> $\dim \mathcal{N}(\mathcal{B}) = 1$

Since $B^p = 0$, $\mathcal{B}$ is represented by

$$B = \begin{pmatrix} 0 & b_1 & b_2 \\ 0 & 0 & b_3 \\ 0 & 0 & 0 \end{pmatrix}$$

with $b_1 \neq 0$ and $b_3 \neq 0$. Then $B^3 = 0$. By condition (1.2) in conjunction with corollary 3.1 of [T-1], we can choose an even positive integer p such that

$$Q_p{}^T A_p{}^{\frac{3}{2}p-2} B_p = \underbrace{B^2AB^2A\ldots B^2AB^2}_{B^2A \text{ appears } \frac{p}{2}-1 \text{ times}} = 0 \tag{3.8}$$

With

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$B^2AB^2 = (b_1b_3)^2 \begin{bmatrix} 0 & 0 & a_{31} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$B^2A = b_1b_3 \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From (3.8) $(b_1b_3)^{\frac{p}{2}} \cdot a_{31}^{\frac{p}{2}-1} = 0$ which implies that $a_{31} = 0$. With $a_{31} = 0$

$$B^2AB^2 = 0$$

and

$$S_{2p-3}(A,B)B = B^2ABAB\ldots AB + BAB^2AB\ldots AB + \ldots + BABA\ldots AB^2 \tag{3.9}$$

The right hand side of (3.9) consists of p-1 terms. Define

$$H \triangleq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} B^2AB &= a_{32}b_1b_3{}^2H \\ HAB &= b_3a_{32}H \\ BAB^2 &= a_{21}b_1{}^2b_3H \\ BAH &= a_{21}b_1G \end{aligned} \tag{3.10}$$

Substitution from (3.10) into (3.9) yields

$$S_{2p-3}(A,B)B = b_1b_3\{(a_{32}b_3)^{p-2} + a_{21}b_1(a_{32}b_3)^{p-3} + \ldots$$

$$\ldots + (a_{21}b_1)^{p-3}a_{32}b_3 + (a_{21}b_1)^{p-2}\}H$$

Set $\alpha \triangleq a_{32}b_3$, $\beta = a_{21}b_1$ and note that $\alpha = \beta = 0$ if and only if $a_{21} = a_{32} = 0$. By condition (1.2)

$$Q_p^T A_p^{2p-3} B_p = S_{2p-3}(A,B)B = b_1 b_3 \sum_{i=0}^{p-1} x^i y^{p-2-i} = 0 \tag{3.11}$$

By corollary 3.1 of [T-1] we also have

$$Q_{p+1}^T A_{p+1}^{2p-1} B_{p+1} = b_1 b_3 \sum_{i=0}^{p-1} x^i y^{p-1-i} = 0 \tag{3.12}$$

Combining (3.11) and (3.12)

$$\sum_{i=0}^{p-1} x^i y^{p-1-i} - x \sum_{i=0}^{p-2} x^i y^{p-2-i} = y^{p-1} = 0$$

so $y = 0$. But then $x = 0$ and, consequently, $a_{21} = a_{32} = 0$.

The characteristic quasipolynomial for system (3.1) is then a polynomial in λ given by

$$\phi(\lambda, e^{-\lambda}) = \det(\lambda I - A - Be^{-\lambda}) = \begin{vmatrix} \lambda - a_{11} & -a_{12} - b_1 e^{-\lambda} & -a_{13} - b_2 e^{-\lambda} \\ 0 & \lambda - a_{22} & -a_{23} - b_3 e^{-\lambda} \\ 0 & 0 & \lambda - a_{33} \end{vmatrix}$$

$$= \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + (a_{11}a_{22} + a_{11}a_{33})\lambda - a_{11}a_{22}a_{33}.$$

IV. SUMMARY

This paper shows that second and third order linear time-delay systems with piecewise constant control inputs are controllable to the zero state in finite time if and only if the characteristic quasipolynomials for their homogeneous systems reduce to polynomials. Hence, the open loop systems have only a finite number of eigenvalues. It is likely that a similar conclusion also holds for higher order systems.

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A RESULT IN THE CONTROL OF SYSTEMS GOVERNED BY LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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ABSTRACT

The problem of controlling the state of a linear time-invariant differential-difference equation system to the zero state of the appropriate function space is considered with control inputs restricted to piecewise constant functions. Such control inputs frequently occur in sampled-data control of processes. Earlier work revealed that an algebraic necessity condition implied that the characteristic quasipolynomial must be a polynomial for all second and third order systems in which the control objective was achieved. It is shown by spectral analysis arguments that this requirement holds true for linear time-invariant single delay systems of any order.

I. INTRODUCTION

Since some physical and biological systems are governed by differential-difference equations, it may be of interest to determine conditions for bringing the state of such systems to the zero state. It is known [1] that a multiple delay system of the form

$$\dot{y}(t) = \sum_{i=0}^{m} A_i \, y(t-ih) \tag{1}$$

with $y(t)\varepsilon R^p$ and $h>0$ can be transformed to a system with a single unit delay

$$\dot{x}(t) = Ax(t) + Bx(t-1) , \quad x(t)\varepsilon R^n \tag{2}$$

where $n = pm$. Therefore this paper will be concerned only with the control of system (2). Control inputs for process (2) are commonly generated based on sampled values of some linear combination of the components of the vector function $x(\cdot)$. The sampling will be assumed to occur at the integer time values $t=0, 1, 2, \ldots,$ and it is further assumed that the control signals remain constant between sampling instants. Note that the state x_t of system (2) at time t is defined as

$$x_t \triangleq x(t+\theta), \quad \theta\,\varepsilon\,[-1, 0]$$

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Hence x_t is an element of a function space, e.g. $\mathscr{C}\left([-1, 0]; R^n\right)$. The specific problem considered can then be restated as follows:

Given the controlled process

$$\dot{x}(t) = Ax(t) + Bx(t-1) + u(t) , \quad t > 0 \tag{3}$$

with arbitrary initial condition $x(t) = \phi(t)$, $\phi \varepsilon \mathscr{C}\left([-1, 0]; R^n\right)$, determine necessary conditions for controlling the system state from $x_0 = \phi$ to the zero state in finite time using piecewise constant control

$$u(t) = u(k) \quad , \quad k \leq t < k+1 \tag{4}$$

for $k = 0, 1, 2, \ldots$.

This problem was briefly considered in [2] where it was shown that there exist time-delay systems for which this type of control is possible. Other prior results, applicable to low-order systems, are reviewed in the next section. In Section III the main result is presented along with an outline of the proof. Specifically, theorem 1 states that it is necessary for system (3) to have a finite spectrum in order to be zero state controllable with control (4). The paper concludes in Section IV with a brief discussion of some implications for the design of zero state sampled-data controllers for process (3).

II. CONTROL OF LOW ORDER SYSTEMS

A convenient representation of the infinite dimensional system (3)-(4) consists of a finite dimensional differential equation and an associated boundary condition. This representational form has been frequently used in the study of pointwise degeneracy of time-delay systems [3,4,5]. Define for $p \geq 1$ the matrices

$$A_p \triangleq \begin{bmatrix} A & & & & 0 \\ B & A & & & \\ & B & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & \cdot & \cdot \\ & & & B & A \end{bmatrix}_{np \times np} \qquad B_p \triangleq \begin{bmatrix} B \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}_{np \times n}$$

and the n-dimensional vectors

$$y_i(t) \triangleq x\big(t+(i-1)\big) \quad , \quad 0\le t\le 1 \quad , \quad i = 1, 2, 3, \ldots \tag{5}$$

$$\psi(t) \triangleq \phi(t-1) \quad , \quad 0\le t\le 1 \, .$$

By continuity $y_{i+1}(0) = y_i(1)$, $i = 1, 2, 3, \ldots$ and (3) - (4) can be replaced by the np-dimensional system equation

$$\dot{Y}_p(t) = A_p Y_p(t) + B_p \psi(t) + U_p \, , \; 0\le t\le 1 \tag{6}$$

together with the two-point boundary condition

$$Y_p(0) = J_p Y_p(1) + \begin{bmatrix} \phi(0) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{7}$$

where

$$Y_p(t) \triangleq \begin{bmatrix} y_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_p(t) \end{bmatrix} , \quad J_p \triangleq \begin{bmatrix} 0 & & & \\ I_n & 0 & & \text{\Large 0} \\ & In & \cdot & \\ \text{\Large 0} & & \cdot & \cdot \\ & & & I_n \quad 0 \end{bmatrix}_{np \times np} , \quad U_p \triangleq \begin{bmatrix} u(0) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u(p-1) \end{bmatrix} .$$

It was shown in [2] that if the state of (3) - (4) was controllable to the zero state at time t_2, then there exists a smallest positive integer $t_1 \le t_2$ such that $x_{t_1} = 0$ for all initial conditions ϕ. This means that with $Q_p^T \triangleq [0 \ldots 0 \; I_n]_{n \times np}$

$$Q_p^T Y_p(t) = 0 \quad \forall \, t\epsilon[0, 1] \tag{8}$$

for some positive integer p if and only if (3)-(4) is zero state controllable. Solving (6)-(7) for $Y_p(t)$ and substituting into (8) leads to an integral representation of condition (8). A necessary condition for (8) to be satisfied is

$$Q_p^T A_p^i B_p = 0 \, , \quad i = 0, 1, 2, \ldots, np-1 \tag{9}$$

as shown in [6]. Since the dimensions of the matrices in (9) increase linearly with the index p (and hence increase with time t), the interpretation of (9) in terms of the original system matrices A and B quickly becomes intractible. For low order system, for which $n = 1, 2,$ or 3, the following results were obtained [7,8]:

<u>n = 1</u>: zero state control given by (4) not possible.

<u>n = 2</u>: zero state control possible only if system (3) has canonical form

$$\dot{x}(t) = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} x(t) + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} x(t-1)$$

with $b \neq 0$. Then the characteristic quasipolynomial $\Delta(\lambda) \triangleq |\lambda I - A - Be^{-\lambda}|$ is a polynomial in λ only and system (3) has a finite spectrum.

<u>n=3:</u> zero state control again possible only if the system spectrum in finite.

It is reasonable to inquire if the same requirement of finite spectra applies to single delay systems of any order. The next section provides an affirmative answer.

III. DERIVATION OF MAIN RESULT

First we define precisely the concept of zero state controllability to be used in the ensuing development.

<u>Definition 1</u>.

System (3) is <u>zero state controllable with piecewise constant input</u> (4) (z.s.c. for short) if there exists a positive integer r such that to each initial function ϕ there exists a piecewise constant control u which transfers the system state from ϕ to zero at $t \leq r$.

<u>Definition 2</u>.

The initial state ϕ of (3) is z.s.c. at time $t \geq 0$ if there exists a piecewise constant u such that $x_t(\phi,u) = 0$.

(Note: $x_t(\phi,u)$ denotes the state of system (3) at time t with control u on $[0,t]$ and initial function ϕ).

Assume next that system (3) is z.s.c. and let $u(\cdot)$ be a control which brings the state to zero at some integer time s. Since $x(t,\phi,u) = 0$ for all t sufficiently large, its Laplace transform $X(\lambda)$ is an entire function [9]. Furthermore

$$X(\lambda)= \frac{\left[\operatorname{adj}(\lambda I-A-Be^{-\lambda})\right]\left[\phi(0) + e^{-\lambda}B\int_{-1}^{0}\phi(\tau)e^{-\lambda\tau}d\tau + \frac{1}{\lambda}(1-e^{-\lambda})\sum_{j=0}^{s-2}u(j)e^{-\lambda j}\right]}{\det(\lambda I-A-Be^{-\lambda})} \tag{10}$$

Whenever ϕ is constant on $[-1,0]$, equation (10) can be viewed as the ratio of two polynomials in the intermediates λ and $\mu \triangleq e^{-\lambda}$. We will show that the number of zeroes common to both the numerator and the denominator polynomials is finite so that their greatest common divisor is a polynomial in λ only.

The locations of the zeroes of the denominator are characterized by a simple extension of a lemma in [10].

Lemma 1.

There exists a real number α such that all solutions of $\det(\lambda I-A-Be^{-\lambda}) = 0$ satisfy $\operatorname{Re}\lambda < \alpha$. Furthermore, there are only a finite number of solutions in any vertical strip in the complex plane.

The asymptotic distribution of the zeroes of the numerator can be determined by dominant term analysis [11] together with the next easily proven lemma.

Lemma 2.

If system (3) is z.s.c., there exists for each ϕ an infinite set S of positive integers such that ϕ is z.s.c. at time s with $u(s-2) \neq 0$, $u(k) = 0 \quad \forall k \geq s-1$, for each $s \in S$.

With $\phi = 1$ as initial function there exists by lemma 2 at least one component of $u(s-2)$ which is non-zero. Without loss of generality we may take the first component of $u(s-2)$ equal to $a \neq 0$. The dominant terms of the first component of the numerator of (4) for large s as $|\lambda| \to \infty$ and $|\mu| \to \infty$ show that the asymptotic location of the numerator zeroes, $z_k = x_k \pm iy_k$, are determined by solving $\lambda = a\mu^{s-1}$. Hence [11, p.35]

$$x_k(s) = \frac{1}{s-1}\left(-\ln y_k(s) + \ln |a(s)|\right) + \varepsilon$$

where $\varepsilon \to 0$ as $k \to \infty$. By lemma 2 the index set S is infinite. Clearly, if the numerator and the denominator polynomials have an infinite number of common zeroes, these zeroes must be asymptotic zeroes of the numerator and their location in the complex plane must be independent of $s \in S$. If $s_1, s_2 \in S$ with $s_1 \neq s_2$ and $y_{k_1}(s_1) = y_{k_2}(s_2) \triangleq y_k$ for positive integers k_1 and k_2, then $z_{k_1}(s_1) = z_{k_2}(s_2)$ only if

$$(s_1 - s_2)\ln y_k = (s_1 - 1)\ln|a(s_2)| - (s_2 - 1)\ln|a(s_1)| + \hat{c} \qquad (11)$$

where $\hat{\varepsilon} \to 0$ as $k \to \infty$. By the properties of y_k[11, p. 36] there exists at most a finite number of integer values k for which (11) is satisfied. Then the number of zeroes common to numerators for different $s \in S$ is finite and, consequently, the numerator and the denominator polynomials in (10) do not possess a common divisor $d(\lambda,\mu)$ where d depends explicitly on μ. Lemma 3 from [12] then gives theorem 1 directly.

Lemma 3.

If $p_1(\lambda,\mu)$ and $p_2(\lambda,\mu)$ are nonzero polynomials, with p_2 not of the form $p_2(\lambda)$, and if

$\dfrac{p_1(\lambda,\mu)}{p_2(\lambda,\mu)}$ is an entire function, then the polynomials $p_1(\lambda,\mu)$ and $p_2(\lambda,\mu)$ have a nonconstant greatest divisor $d(\lambda,\mu)$ which is not of the form $d(\lambda)$.

Theorem 1.

If system (3) is z.s.c., then the characteristic quasipolynomial $|\lambda I - A - Be^{-\lambda}|$ is a polynomial in λ only.

IV. IMPLICATIONS FOR CONTROLLER DESIGN

Extension of theorem 1 to the case of a single delay of length h seems immediate. Theorem 1 is then valid for control signals which are piecewise constant over any interval of rational length.

By [13] all second order z.s.c. time-delay systems (3) are pointwise complete since their A and B matrices commute. Consequently, the homogeneous part of (3) in case $n = 2$ is governed by an ordinary differential equation for $t > 1$ [14, 15]. This property, which may also apply to higher order systems, suggests the possibility of designing relatively simple sampled-data feedback controllers for systems (3).

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Zero State Control of Time-Delay Systems with Piecewise Constant Inputs

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Abstract

It is shown that a necessary condition for controlling the state of the differential-delay system $\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i x(t-i) + Cu(t)$, $u(t) = u_k$ for $t \in (k, k+1]$, $k = 0,1,2,\ldots$ to the zero state is that the characteristic quasipolynomial $\Delta(\lambda) = |\lambda I - A - \sum_{i=1}^{m} B_i e^{-\lambda i}|$ is a polynomial in λ only.

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1. Introduction

In feedback control of industrial processes with linear and nonlinear dynamics the control inputs are often piecewise constant functions of time. In this paper control of time-invariant systems governed by linear differential-difference equations

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i x(t-i) + Cu(t), \ t > 0, \ B_m \neq 0 \qquad (1)$$

is considered with piecewise constant inputs given by

$$u(t) = u_k, \quad t \in (k,k+1], \quad k = 0,1,2,\ldots \qquad (2)$$

Let $\mathcal{C}([-m,0];R^n)$ denote the function space consisting of all continuous maps $[-m,0] \to R^n$. The objective is to control the state x_t of system (1), defined as $x_t \overset{\Delta}{=} \{x(t+\theta), -m \leq \theta \leq 0\}$, to the origin of its state space $\mathcal{C}([-m,0];R^n)$. We seek to derive necessary structural conditions for such control. In particular it will be shown that $\det(\lambda I - A - \sum_{i=1}^{m} B_i e^{-i\lambda})$ must be a polynomial in λ. That (1) must have a finite spectrum to be zero state controllable generalizes some earlier results [9,10] for second and third order systems.

The organization of the paper is as follows: Section 2 contains definitions and lemmas. Section 3 presents the spectral considerations which lead to the main result for systems with a single delay of length m, and section 4 extends this result to the more general time-delay systems of form (1) with several commensurable delays.

2. Single Delay System

Consider the linear time-invariant, single delay system

$$\dot{z}(t) = \hat{A}z(t) + \hat{B}z(t-m) + u(t), \quad t > 0, \quad \hat{B} \neq 0 \tag{3}$$

with $z(t)$, $u(t) \in R^p$ and with continuous initial condition $z(\tau) = \phi(\tau)$, $\tau \in [-m,0]$. The admissible control functions are given by (2). The following notation will be used: $z_t \triangleq \{z(t+\theta), -m \leq \theta \leq 0\}$ is the state of system (3) at time t, and $z_t(\phi,u)$ is the state at time t resulting from control u on $(0,t]$ and initial condition $z_o = \phi$. We first seek to determine necessary conditions for bringing the state of system (3) to the zero state ($z_t = 0$) in finite time using piecewise constant controls. Such conditions are of interest for sampled-data control of time-delay systems. The connection between systems (1) and (3) will be described in section 4.

Definition 1.

System (3) is zero state controllable with piecewise constant inputs of form (2) (z.s.c. for short) if there exists a positive real number r such that for each initial function $\phi \in C([-m,0];R^p)$ a piecewise constant control u can be found which controls the system to its zero state in time less than or equal to r.

To derive the main result of this paper, it will also be convenient to refer to a particular state as being zero state controllable.

Definition 2.

The initial state $\phi \in C([-m,0];R^p)$ of (3) is z.s.c. in time τ ($\tau \geq 0$)

if there exists a piecewise constant u (restricted by (2)) such that $z_\tau(\phi,u) = 0$.

Remark: The definitions above also apply to system (1) when z_t and p are replaced by x_t and n, respectively.

If system (3) is z.s.c., definition 1 implies that each initial state ϕ of (3) is z.s.c. in some time $k \leq r$. In proving the main result use will be made of a subset $\hat{C}$ of $C([-m,0];R^p)$ consisting of all (initial) functions which never yield the zero state when $u \equiv 0$; i.e.,

$$\hat{C} \triangleq \{\phi \in C([-m,0];R^p) \mid z_t(\phi,0) \neq 0 \text{ for all } t > 0\} .$$

The set $\hat{C}$ can also be defined as the complement in $C([-m,0];R^p)$ of

$$C \triangleq \{\phi \in C([-m,0];R^p) \mid \phi(0) = 0 \text{ and}$$

$$\phi(\theta) \in N(\hat{B}) \ \forall\ \theta \in [-m,0]\} .$$

The following lemma gives a partial yet sufficient characterization of the control which for each $\phi \in \hat{C}$ transfers the state to zero in minimum time.

Lemma 1.

Suppose system (3) is z.s.c. For each $\phi \in \hat{C}$ there exists a smallest positive time k such that $z_k(\phi,u) = 0$ for some admissible control u. Furthermore, k is an integer and the control u satisfies $u_{k-i} = 0$, $i = 1,2,\ldots,m$, and $u_{k-m-1} \neq 0$.

Remark: It is easily shown that if $\phi \in \hat{C}$ is z.s.c. at some time t, then $t \geq m+1$. Hence the subscript k-m-1 in the statement of the lemma is nonnegative.

Proof.

Let τ be the smallest value of t for which $z_t(\phi,u) = 0$ for any admissible control. Then $\tau \varepsilon (\ell,\ell+1]$ for some positive integer ℓ. From (3) and the continuity of $z(t)$ it follows that $\hat{B}z(t) = 0$ on $[\ell + 1 - 2m,\tau]$ and that $u_\ell \equiv u_{\ell-1} = \dots = u_{\ell+1-m} = 0$. Will show that τ is an integer. Suppose, on the contrary, that $\tau \varepsilon (\ell,\ell+1)$. Then $\hat{B}z(t-m) + u_{\ell-m} = 0 \;\forall\; \tau \varepsilon (\tau-m, \ell+1-m]$. By continuity $u_{\ell-m} = \hat{B}z(\ell+1-2m) = 0$ so $\hat{B}z(t) = 0$ on $[\tau-2m,\tau]$. Since $z_{\tau-m} \neq 0$, $\hat{B}$ must be singular. Let $z^I \varepsilon N(\hat{B})$ and $z^{II} \varepsilon N^\perp(\hat{B})$. Then

$$\begin{bmatrix} \dot{z}^I(t) \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z^I(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} z^I(t-m) \\ z^{II}(t-m) \end{bmatrix}$$

for $t \varepsilon (\ell-m, \ell+1-m)$. Since B_{22} is nonsingular, $\dot{z}^I(t) = [A_{11} - B_{12}B_{22}^{-1}A_{12}]\, z^I(t)$ with $z^I(\tau-m) = 0$ which implies $z(\ell-m) = 0$ and $z_\ell = 0$ (contradiction). Hence the shortest possible time for any state to reach the zero state is an integer value. For the given ϕ let k be this minimal time. Then $z^{II}(t) \equiv 0$ on $[k-2m,k]$ and $u_{k-i} = 0$, $i = 1,2,\dots,m$. For $t \varepsilon (k-m-1,k-m)$ $A_{21}z^I(t) + B_{22}z^{II}(t-m) + u^{II}_{k-m-1} = 0$ so $u^{II}_{k-m-1} = 0$. Then $\dot{z}^I(t) = (A_{11} - B_{12}B_{22}^{-1}A_{21})\, z^I(t) + u^I_{k-m-1}$ with $z^I(k-m) = 0$ and $z^I(t) \neq 0$ for some $t \varepsilon (k-m-1,k-m]$. Hence $u_{k-m-1} \neq 0$. Finally, $k \leq r$ follows from definition 1.

Lemma 2.

If system (3) is z.s.c., there exists for each $\phi \varepsilon \hat{C}$ an infinite set S of positive integers s with corresponding admissible controls $u^{(s)}$

(indexed by s) such that for each $s \in S$:

(i) $z_s(\phi, u^{(s)}) = 0$ and $z_t(\phi, u^{(s)}) \neq 0 \;\forall\; t < s$

(ii) $u^{(s)}_{s-m-1} \neq 0$ and $u^{(s)}_{s-i} = 0$, $i = 1, 2, \ldots, m$.

Proof.

Let i be any positive integer. With $\phi \in \hat{C}$ as initial state at time $t = 0$, apply no control on $(0, ir]$ where r is as given in definition 1. Then, by definition of the set $\hat{C}$, $z_{ir} \in \hat{C}$. Since the system is both z.s.c. and time-invariant, there exist by lemma 1 a smallest integer k_i, $m+1 \leq k_i \leq r$, and a piecewise constant control $\hat{u}_j$, $j = 0, 1, \ldots, k_i - 1$, which brings the state z_{ir} at time ir to the zero state at time $ir + k_i$ with $\hat{u}_{k_i - \ell} = 0$, $\ell = 1, 2, \ldots, m$ and $\hat{u}_{k_i - m - 1} \neq 0$. To complete the proof set $S \triangleq \{ir + k_i,\ i = 1, 2, \ldots\}$ and

$$u^{(s)}_j \triangleq \begin{cases} 0 & \text{for } j = 0, 1, \ldots, ir-1 \\ \hat{u}_{j-ri} & \text{for } j = ir, \ldots, ir + k_i - 1 \end{cases}$$

The facts that S contains more than one element and that the elements of S can be chosen arbitrarily large play fundamental roles in proving the main result.

3. Spectral Arguments

Assume that system (3) is z.s.c. and let $\phi \in \hat{C}$ be the initial condition at $t = 0$. Choose an $s \in S$ and the corresponding control $u = u^{(s)}$ described in lemma 2. Then $z_s(\phi, u) = 0$. The Laplace transform of the differential-difference equation is then

$$z(\lambda) = \frac{[\operatorname{adj}(\lambda I-\hat{A}-\hat{B}e^{-\lambda m})][\phi(0)+e^{-\lambda m}\hat{B}\int_{-m}^{0}\phi(\tau)e^{-\lambda\tau}d\tau + \frac{1}{\lambda}(1-e^{-\lambda})\sum_{j=0}^{s-m-1}u_j e^{-\lambda j}]}{\det(\lambda I-\hat{A}-\hat{B}e^{-\lambda m})} \tag{4}$$

Since $z(t) = 0$ for all t sufficiently large, $z(\lambda)$ is an entire function [1]. It is convenient to define $\mu \triangleq e^{-\lambda}$ and to consider (4) as a ratio of two polynomials in the indeterminates λ and μ. This is e.g. possible whenever ϕ is constant on $[-m,0]$.

We will show that the number of zeroes common to both the numerator and the denominator polynomials is finite. Their greatest common divisor is then a polynomial in λ only. The distribution of denominator zeroes is determined by slight extension of a result in [2, p.18].

Lemma 3.

There exists a real number α such that all solutions of $\Delta(\lambda,e^{-\lambda}) = \det(\lambda I-\hat{A}-\hat{B}e^{-\lambda m}) = 0$ satisfy $\mathrm{Re}\lambda < \alpha$. Furthermore, there are only a finite number of solutions in any vertical strip in the complex plane.

Hence the denominator zeroes are independent of the choice of $s \in S$. Consider next the distribution of zeroes of the numerator. Let the initial condition be the constant function $\phi(\cdot) = 1(\cdot) \in \hat{C}$. When ϕ is z.s.c. at time s, there exists by lemma 2 at least one component of u_{s-m-1} which is nonzero. Without loss of generality let the first component of u_{s-m-1} equal $a \neq 0$. Then

$$z(\lambda,\mu) = \frac{[\mathrm{adj}(\lambda I-\hat{A}-\hat{B}\mu^m)]\left\{\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\lambda + \hat{B}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}(1-\mu^m) + (1-\mu)\sum_{j=0}^{s-m-1} u_j\mu^j\right\}}{\lambda \det(\lambda I-\hat{A}-\hat{B}\mu^m)} \tag{5}$$

Note that

$$\mathrm{adj}(\lambda I-\hat{A}-\hat{B}\mu^m) \triangleq I\lambda^{n-1} + B_1(\mu^m)\lambda^{n-2} + B_2(\mu^m)\lambda^{n-3} + \dots$$

$$\dots + B_{n-1}(\mu^m)$$

where $B_i(\mu^m)$ is a polynomial matrix of order $\leq i$ in μ^m. When $|\lambda| \to \infty$, $|\mu| \to \infty$ and for s sufficiently large, the dominant terms of the first component of the numerator reduce to

$$\lambda^n - a\lambda^{n-1}\mu^{s-m} + \sum_{j=1}^{n-1} \lambda^{n-1-j}p_j(\mu) \tag{6}$$

where $p_j(\mu)$ is a polynomial of order $\leq s + (j-1)m$ in μ. For some choice of constants β_j the sum $\sum_{j=1}^{n-1} \lambda^{n-1-j}p_j(\mu)$ is dominated by $\sum_{j=1}^{n-1} \beta_j\lambda^{n-1-j}\mu^{s+(j-1)m}$. Will show that the asymptotic distribution of zeroes of

$$\lambda^n - a\lambda^{n-1}\mu^{s-m} + \sum_{j=1}^{n-1} \beta_j\lambda^{n-1-j}\mu^{s+(j-1)m} \tag{7}$$

is found by solving $\lambda = a\mu^{s-m}$. Hence, the asymptotic locations of zeroes of (5) (and of (6)) are also determined by $\lambda = a\mu^{s-m}$. Set (7) equal to zero and rewrite as

$$\frac{\lambda}{u^m} = \mu^{s-2m}\left\{a - \sum_{j=1}^{n-1} \beta_j \left(\frac{\mu^m}{\lambda}\right)^j\right\}$$

which shows that $|\frac{\lambda}{\mu^m}| \to \infty$ as $|\lambda| \to \infty$ and $|\mu| \to \infty$ for s sufficiently large. Hence, the dominating terms in (5), (6), and (7) are $\lambda^n - a\lambda^{n-1}\mu^{s-m}$ and the asymptotic locations of zeroes are found by solving $\lambda = a\mu^{s-m}$. The asymptotic zeroes $z_k(s) = x_k(s) \pm iy_k(s)$ for $s \in S$ are given by [3, p.35 with $p = 1$, $\tau = s-m$].

$$x_k(s) = \frac{1}{s-m} (-\ell n\ y_k(s) + \ell n|a(s)|) + \varepsilon$$

when $\varepsilon \to 0$ as $k \to \infty$. Let s_1, $s_2 \in S$ be distinct and suppose $y_{k_1}(s_1) = y_{k_2}(s_2) \triangleq y_k$ for positive integers k_1 and k_2. For the zeroes $z_{k_1}(s_1)$ and $z_{k_2}(s_2)$ to be identical the real parts must also coincide. This requires

$$(s_1 - s_2)\ \ell n\ y_k = (s_1-m)\ \ell n\ |a(s_2)| - (s_2-m)\ \ell n\ |a(s_1)| + \hat{\varepsilon} \tag{8}$$

where $a(s_i)$, $i=1,2$, are independent of k and $\hat{\varepsilon} \to 0$ as $k \to \infty$. Since $s_1 \neq s_2$ and since $\ell n\ y_k$ is a strictly monotone function of k [3, p.36], there exists at most a finite number of integer values k for which (8) is satisfied. Hence the set of common zeroes of numerator polynomials for different $s \in S$ is finite. For an infinite number of zeroes to be common to both the denominator and the numerator for a fixed $s \in S$, an infinite number of asymptotic zeroes of the numerator polynomial must coincide with zeroes of the denominator by lemma 3. Since the zeroes of the denominator are fixed (i.e., independent of $s \in S$) and since at most a finite number of zeroes are common to numerator polynomials for different values of $s \in S$, the numerator and denominator polynomials in (5) do not possess a common divisor of the form $d(\lambda,\mu)$, where d depends explicitly on μ, for more than one value $s \in S$ sufficiently large.

Lemma 4. [4, lemma 3.2]

Let $p(\lambda,\mu)$ and $p_o(\lambda,\mu)$ be nonzero polynomials such that $\frac{p(\lambda,\mu)}{p_o(\lambda,\mu)}$, $\mu = e^{-\lambda}$, is an entire function. If p_o is not of the form $p_o(\lambda)$, then the polynomials $p(\lambda,\mu)$ and $p_o(\lambda,\mu)$ have a nonconstant greatest divisor $d(\lambda,\mu)$ which is not of the form $d(\lambda)$.

Lemma 4 applied to (5) with $s \neq \hat{s}$ shows that $p_o = \Delta(\lambda,e^{-\lambda})$ must be a polynomials in λ only. We have then proved:

Theorem 1.

If $\dot{z}(t) = \hat{A}z(t) + \hat{B}z(t-m) + u(t)$ is z.s.c., then $\det(\lambda I-\hat{A}-\hat{B}e^{-\lambda m})$ is a polynomial in λ.

4. Multiple Delay System

Consider the multiple delay system

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i x(t-i) + Cv(t) \tag{9}$$

where $x(t) \in R^n$, $v(t) \in R^w$ and $v(t)$ is a piecewise constant function restricted as in (2). Let

$$\hat{A} \triangleq \begin{bmatrix} A & B_1 & \cdot & \cdot & \cdot & B_{m-1} \\ & A & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & \text{O} & & & \cdot & B_1 \\ & & & & & A \end{bmatrix}, \quad \hat{B} \triangleq \begin{bmatrix} B_m & & & & & \\ B_{m-1} & B_m & & & \text{O} & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ B_1 & \cdot & \cdot & \cdot & B_{m-1} & B_m \end{bmatrix},$$

$$u \triangleq \begin{bmatrix} Cv(t) \\ Cv(t-1) \\ \cdot \\ \cdot \\ \cdot \\ Cv(t-m+1) \end{bmatrix}$$

System (9) can be rewritten as a single delay system

$$\dot{z}(t) = \hat{A}z(t) + \hat{B}z(t-m) + u(t) \tag{10}$$

of order $p = nm$ with $z^T(t) = [x^T(t), x^T(t-1), \ldots, x^T(t-m+1)]$. Clearly (9) is z.s.c. only if (10) is z.s.c. Furthermore, if $e^{\lambda t}x$ is a solution to the homogeneous part of (9), then $e^{\lambda t}z$ with $z^T = [x^T, e^{-\lambda}x^T, \ldots, e^{-\lambda(m-1)}x^T]$ is a solution to the homogeneous part of (10).

Theorem 2.

A necessary condition for controlling the state x_t of $\dot{x}(t) = Ax(t)$

$+ \sum_{i=1}^{m} B_i x(t-i) + Cu(t)$ to the zero state with controlls (2) is that

$\Delta = |\lambda I - A - \sum_{i=1}^{m} B_i e^{-i\lambda}|$ is a polynomial.

Proof

Zero state controllability of (9) requires z.s.c. of (10) so by theorem 1 $\det(\lambda I - \hat{A} - \hat{B}e^{-\lambda m})$ must be a polynomial. Since each root of the characteristic equation for (9) satisfies $\det(\lambda I - \hat{A} - \hat{B}e^{-\lambda m}) = 0$, the set of roots of $\Delta = |\lambda I - A - \sum_{i=1}^{m} B_i e^{-\lambda i}| = 0$ is finite and hence Δ is a polynomial in λ only.

5. Conclusion

It was shown that the state of the time-delay system $\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i x(t-i) + Cu(t)$ could be brought to zero with piecewise constant control signals only if the system has a finite spectrum. A similar reduction of the quasipolynomial to a polynomial has been noted in some pointwise degenerate systems [5,6]. However, zero state controllability is not dependent on pointwise degeneracy. The latter property cannot occur in system $\dot{x}(t) = Ax(t) + Bx(t-1)$ when $n = 2$ or when A and B commute [7], yet the single delay system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix} + \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

is controllable to the zero state with piecewise constant control (2) [8].

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COMPUTER CONTROL FOR LOW ORDER TIME-DELAY SYSTEMS

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Abstract

Sampled-data control of low order continuous-time dynamic systems with time delays is considered. A class of controllers which will transfer the state of the infinite dimensional delay system to the zero state in finite time is determined. The control algorithms require very little data storage and are easily implemented on any digital computer.

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COMPUTER CONTROL FOR LOW ORDER TIME-DELAY SYSTEMS

Arild Thowsen

I. Introduction

In this paper we consider digital feedback control of continuous-time dynamic systems with time delays. Time delays in the process model may result from mass or energy transfers, transportation lags and transmission delays, so there is a fairly wide class of dynamic systems which can be modeled by

$$\dot{x}(t) = Ax(t) + Bx(t-1) + u(t), \quad t > 0 \tag{1}$$

with $x(t) \varepsilon R^n$. A and B are constant $n \times n$ matrices with $B \neq 0$. A system with multiple delays can be described by (1) with a single normalized delay [1]. The homogeneous part of equation (1) is a system of differential-difference equations whose behavior has been extensively studied (see e.g. [2]). This paper, concerned with the control of process (1), focusses on the inhomogeneous equation. It is easily seen that setting $u(t) = 0$ on $[t_1, \infty]$ when $x(t_1) = 0$ does not generally cause $x(t) = 0$ for all $t \geq t_1$ unless $x(t)$ is also zero on the whole interval $[t_1 - 1, t_1]$. Hence an interesting problem is the determination of control inputs which will drive the state x_t of (1) defined by

$$x_t(\theta) = x(t+\theta), \quad -1 \leq \theta \leq 0 \tag{2}$$

to the origin of the chosen state space, e.g., $C([-1, 0]; R^n)$. ($C([-1, 0]; R^n)$ is the space of all continuous functions mapping

$[-1,0]$ into R^n.)

Digital feedback control of dynamic systems consists in parts of generating piecewise constant control signals based on sampled values of $x(t)$. Without significant loss of generality the sampling period for system (1) may be taken to be unity for the problem being considered.

In this paper we will design simple sampled-data feedback controllers for the frequently occuring low order models for which $n = 1$, 2, or 3. The controllers should be able to drive the state of system (1) from any arbitrary initial state to the zero state in a short time. Furthermore, the control algorithm should require little data storage and be easily implementable on a digital processor. Since the feedback solution to the standard linear-quadratic regulator problem for (1) requires infinite data storage and is not capable of bringing x_t to the zero state in finite time, these conditions are seemingly difficult to satisfy.

The organization of the paper is as follows: Section 2 gives necessary conditions on the system structure for the state x_t to be controlled to zero in finite time. In section 3 we derive the algorithm for the class of sampled-data feedback controllers which satisfy the requirements stated above. Section 4 gives a computational example of the control algorithm applied to an unstable second order system with time-delay.

II. Necessary Structural Conditions

Consider the normalized single-delay system (1) with a sampled-data controller generating an input signal based on values of x

measured at each unit time. The control signal is then constant over each individual time interval $[k, k+1]$. To derive necessary conditions for the system to be controlled to the zero state one can study, as in [3] - [6], the behavior of (1) subject to piecewise constant inputs given by

$$u(t) = u(k), \quad k \leq t < k+1, \quad k = 0, 1, 2, \ldots \qquad (3)$$

The following results are pertinent to the present study:

(i) First order system [4].

No control u of the form (3) exists which will transfer the state of the scalar system $\dot{x}(t) = ax(t) + bx(t-1) + u(t)$ (with $b \neq 0$) to the zero state in finite time.

(ii) Second order system [4, 6].

Those second order systems $\dot{x}(t) = Ax(t) + Bx(t-1) + u(t)$ whose state can be brought to zero using control (3) must satisfy the algebraic conditions

$$B^2 = 0, \quad AB = \gamma_1 B, \quad BA = \gamma_2 B, \qquad \gamma_1, \gamma_2 \in R$$

Canonical representation for the homogeneous system is given by

$$\dot{x}(t) = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} x(t) + \begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix} x(t-1) \qquad (4)$$

and the minimal time in which every initial state $x_0(\theta) = \phi(\theta)$, $\theta \in [-1, 0]$, $\phi \in C\left([-1, 0]; R^2\right)$ can be transferred to zero is three time units.

(iii) Third order systems [6]

For those third order systems which are controllable to the zero state by control (3) there are two canonical representations of the homogeneous system dependent on the rank of matrix B. If rank B = 1, then

$$\dot{x}(t) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix} x(t-1) \tag{5}$$

with the additional conditions

$$b_{13}\, a_{31} + b_{23}\, a_{32} = 0$$

$$b_{13}\, a_{32}\, a_{21} + b_{21}\, a_{31}\, a_{12} + b_{13}\, a_{31}\, a_{11} + b_{23}\, a_{32}\, a_{22} = 0$$

If, on the other hand, rank B = 2, then

$$\dot{x}(t) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} x(t) = \begin{pmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix} x(t-1) \tag{6}$$

and the minimal time for transferring any initial state to the zero state is four time units.

These results were all obtained using a "split system" representation [7, 8, 9] of system (1) - (3) and by interpreting an algebraic condition found to be necessary for controlling x_t to zero. A common feature of cases (i) - (iii) is that the characteristic quasipolynomial det $(\lambda I - A - Be^{-\lambda})$ must be a polynomial in λ only for the desired state transfer to be possible. In other words, the system spectrum must be finite. This fact is utilized in the next section to determine digital feedback controllers which quickly bring x_t of system (1) to the zero state.

III. Feedback Controllers

Before developing the class of proposed digital controllers we need to review the meaning of pointwise degeneracy and pointwise completeness for the n^{th} order time-delay systems

$$\dot{x}(t) = Ax(t) + Bx(t-1) , \quad t > 0 \tag{7}$$

Definition

System (7) is pointwise degenerate if there exists a nonzero vector $\eta \varepsilon R^n$ and a time $t_1 > 0$ such that for all solutions $x(\cdot)$ to (7) $\eta^T x(t) = 0$ whenever $t \geq t_1$. Otherwise system (7) is pointwise complete.

The following result can be found in [7] and [8].

Theorem 1.

If system (7) is pointwise complete and has a finite spectrum, there exists a finite dimensional system

$$\dot{x}(t) = Dx(t) \tag{8}$$

of order n which is equivalent to (7) in the sense that any solution $x(\cdot)$ of (7) satisfies (8) for $t \geq n-1$.

To apply theorem 1 it is necessary to show that systems (4), (5) and (6) are pointwise complete.

Theorem 2 [9]

A second order system of form (7) is always pointwise complete.

For third order systems it is known [10, theorem 4] that rank $B = 2$ is necessary for pointwise degeneracy. System (5) is therefore pointwise complete.

Theorem 3 [10]

Any pointwise degenerate third order system of form (7) can be written as

$$\dot{x}(t) = Ax(t) + (AZ - ZA) \; x(t-1)$$

where

$$Z = rq^T e^A$$

and q and r are two vectors which satisfy the equations

$$q^T r = 1 \tag{9}$$

$$q^T e^A r = 0 \tag{10}$$

$$q^T e^A Ar = 0 \tag{11}$$

Theorem 3 requires that $Br = (AZ - ZA)r = Arq^T e^A r - rq^T e^A Ar = 0$ if system (6) is pointwise degenerate. Since $b_{12} \neq 0$ and $b_{23} \neq 0$, r must be of the form $r^T = (\hat{r}, 0, 0)$. By (9) $q^T = \left(\hat{r}^{-1}, \; c_1, \; c_2\right)$ where c_1 and c_2 are arbitrary real numbers. But then $q^T e^A r = e^{a_{11}} \neq 0$ which contradicts (10). Consequently system (6) is pointwise complete.

In deriving the control algorithm for the sampled-data controller the past behavior of the time-delay system must be taken into account. This naturally leads to a time-dependent controller. Furthermore it should be observed that the dynamics of system (1) is not described by

$\dot{x}(t) = Dx(t) + u(t)$ for $t > n-1$, not even in the case of zero input prior to $t = n-1$. This is in spite of the fact that by theorem 1 the homogeneous system is governed by (8) whenever $t > n-1$.

Consider first the second order system (1) with system matrices given by (4). The fundamental matrix $K(t)$ for system (1) satisfies [2]

$$\begin{aligned} &\dot{K}(t) = AK(t) + BK(t-1)\ , \quad t > 0 \\ &K(0) = I \\ &K(t) = 0\ , \quad t\varepsilon[-1,\ 0) \end{aligned} \tag{12}$$

Then

$$x(t) = K(t)\ x(0) + \int_{-1}^{0} K(t-s-1)\ Bx\ (s)ds + \int_{0}^{t} K(t-s)\ u(s)ds \tag{13}$$

On the other hand, since the homogeneous system is pointwise complete and described by (8), the homogeneous solution is

$$x_h(t) = e^{D(t-1)}\left[K(1)x(0) + \int_{-1}^{0} K(-s)\ Bx(s)\ ds\right] \tag{14}$$

for $t \geq 1$. By comparison of (13) and (14),

$$K(t) = e^{D(t-1)}e^{A} \quad \text{for } t \geq 1. \tag{15}$$

A sampled-data feedback algorithm of the form

$$u(t) = \begin{cases} Lx(0) & 0 \leq t < 1 \\ M_1\ x(1) + M_2\ x(0) & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases} \tag{16}$$

will be developed such that $x(t) = 0$ for all $t \geq 3$ irrespective of the initial data. For $t \geq 3$

$$x(t) = e^{D(t-1)}[x(1) - \int_0^1 K(1-s)ds\ u(0)] + \int_0^t K(t-s)\ u(s)\ ds$$
$$= e^{D(t-1)}[x(1) - \int_0^1 e^{A(1-s)}ds\ Lx(0)$$
$$+ \int_0^1 e^{-Ds}dse^A\ Lx(0) + \int_1^2 e^{-Ds}dse^A u(1)] \tag{17}$$

by (15).

Hence $x(3) = 0$ if

$$u(1) = M_1\ x(1) + M_2\ x(0) \tag{18}$$

where

$$M_1 \triangleq -e^{-A}\left(\int_1^2 e^{-Ds}ds\right)^{-1} \tag{19}$$

$$M_2 \triangleq M_1 \int_0^1 [e^{-Ds} - e^{-As}]ds\ e^A\ L \tag{20}$$

The existence of the inverse matrix in (19) follows from the following lemma.

Lemma 1.

The $n \times n$ matrix $H \triangleq \int_a^b e^{Ft}\ dt$ is nonsingular for every matrix F and all real numbers $a < b$.

Proof.

There exists a nonsingular matrix P such that PFP^{-1} is in the Jordan form. Then

$$H = \int_a^b e^{Ft}\ dt = P^{-1} \int_a^b e^{PFP^{-1}t}\ dt\ P$$

and

$$\det\left(\int_a^b e^{PFT^{-1}t}\ dt\right) = \prod_{i=1}^{n} \theta_i$$

$$\text{where } \theta_i = \begin{cases} \dfrac{e^{\lambda_i b} - e^{\lambda_i a}}{\lambda_i} & \text{if } \lambda_i \neq 0 \\ b - a & \text{if } \lambda_i = 0 \end{cases}$$

and λ_i, $i = 1, 2, \ldots, n$; are the eigenvalues of F. Hence, H is nonsingular.

Equation (17) shows that control (18) makes $x(t) \equiv 0$ on $[3,\infty)$. The freedom in choosing matrix L can be used to further shape the transient response of the closed loop system.

Consider next a third order system (1) with A and B matrices given by (6). Similarly to the second order case, comparison of representations for the homogeneous solution yields

$$K(t) = e^{D(t-2)} K(2) \qquad t \geq 2 \tag{21}$$

With control signals

$$u(t) = \begin{cases} Nx(0) & 0 \leq t < 1 \\ P_1x(1) + P_2x(0) & 1 \leq t < 2 \\ R_1x(2) + R_2x(1) + R_3x(0) & 2 \leq t < 3 \\ 0 & 3 \leq t \end{cases} \tag{22}$$

the solution to (1) for $t \geq 5$ becomes

$$\begin{aligned} x(t) &= e^{D(t-2)}[x(2) - \int_0^2 K(2-s)\, u(s)\, ds] + \int_0^3 K(t-s)\, u(s)\, ds \\ &= e^{D(t-2)}\Big\{x(2) - \int_0^1 [K(2-s) - e^{-Ds}K(2)]\, ds\, u(0) \\ &\quad + \int_0^1 [e^{-D(s+1)}K(2) - e^{As}]\, ds\, u(1) + \int_2^3 e^{-Ds} ds\, K(2)\, u(2)\Big\} \end{aligned}$$

Hence the control (22) with

$$R_1 \triangleq - K(2)^{-1} \left(\int_2^3 e^{-Ds} ds \right)^{-1}$$

$$R_2 \triangleq R_1 \int_0^1 [e^{-D(s+1)} K(2) - e^{As}] \, ds \, P_1$$

$$R_3 \triangleq R_1 \left\{ \int_0^1 [e^{-D(s+1)} K(2) - e^{As}] \, ds \, P_2 - \int_0^1 [K(2-s) - e^{-Ds} K(2)] \, ds \, N \right.$$

makes $x(t) \equiv$ on $[5,\infty]$. The existence of R_1 follows from lemmas 1 and 2.

Lemma 2.

K(2) is nonsingular for system (6).

Proof.

From (12) $K(2) = e^{2A} + \int_1^2 e^{A(2-t)} B e^{A(t-1)} dt$. In the canonical form (6) the matrix e^{2A} is upper triangular and nonsingular while the integrand is upper triangular with zero diagonal entries. Hence, $\det\left(K(2)\right) = \det\left(e^{2A}\right) \neq 0$.

The feedback matrices N, P_1, and P_2 in (22) can be chosen to shape the transient response of the closed-loop sampled-data system.

IV. Computational Example

This section illustrates the effect of applying the control algorithm (16) to the unstable second order time-delay system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix} + u(t) \tag{23}$$

which can be transformed into form (4) by a similarity transformation. The system is pointwise complete by theorem 2 and for $t > 1$ the dynamics of the homogeneous system (23) is equivalent (in the sense of theorem 1) to that of

$$\dot{x}_1(t) = (3+e^{-1})\, x_1(t) - (2+e^{-1})\, x_2(t)$$

$$\dot{x}_2(t) = (2+e^{-1})\, x_1(t) - (1+e^{-1})\, x_2(t)$$

With

$$L = \begin{pmatrix} 2.089 & -1.589 \\ 0.555 & -1.055 \end{pmatrix}$$

the other feedback matrices determined by (19) and (20) become

$$M_1 = \begin{pmatrix} -3.730 & 2.148 \\ -2.148 & 0.566 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0.641 & -0.223 \\ 0.641 & -0.223 \end{pmatrix}$$

Figures 1a and 1b show the closed-loop system trajectory for the initial conditions $x_1(\tau) = 10$, $x_2(\tau) = 15$, $\tau\varepsilon[-1,0]$. For $t \geq 3$ $x(t) = 0$.

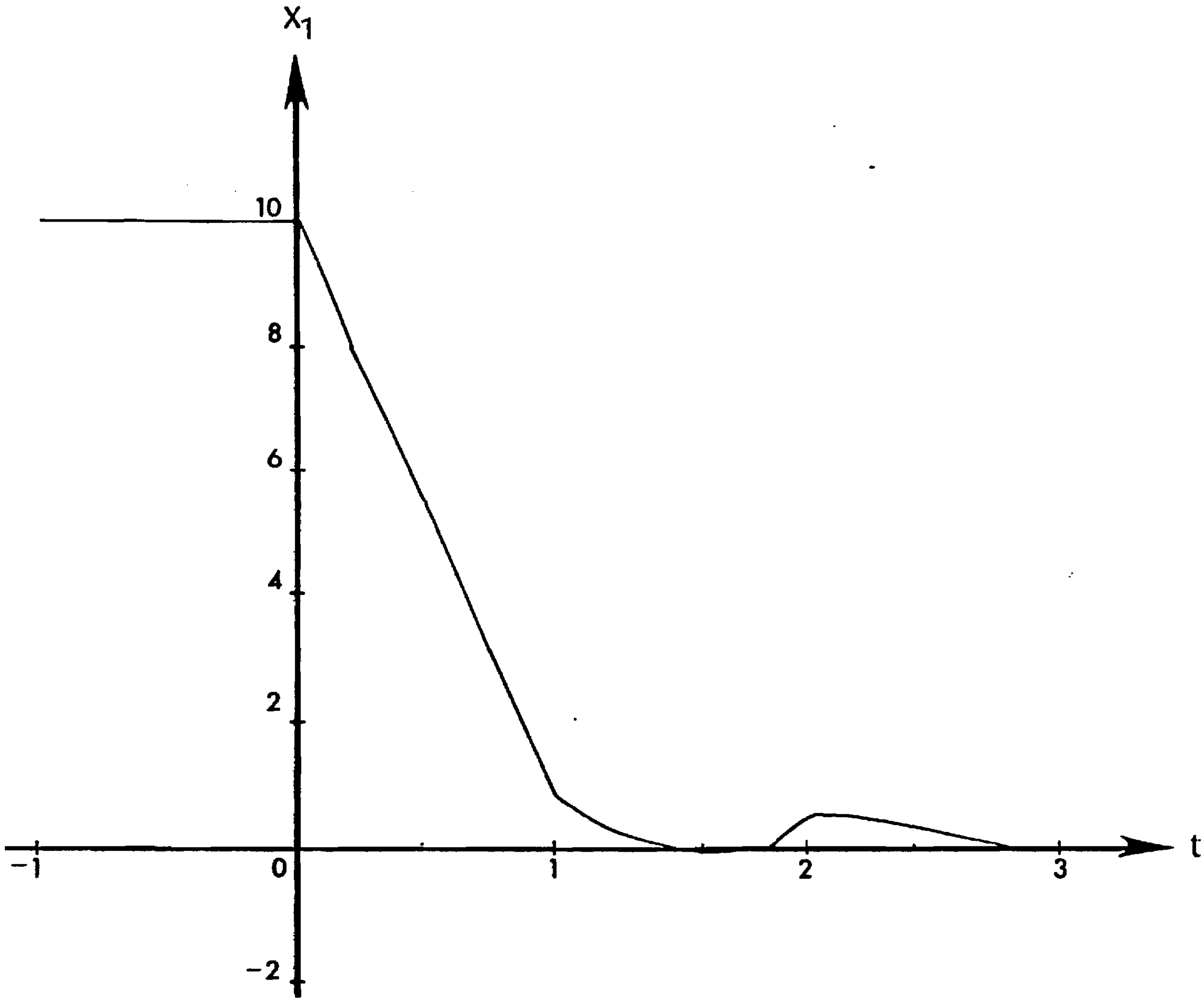
V. Conclusion

A class of time-delay systems which are controllable to the zero state is shown to have finite spectra. Sampled-data feedback controllers are developed for second and third order systems. The new controllers transfer the state of the time-delay system to the zero state in a short time. Due to spectral finiteness the control algorithm requires only a small amount of data storage while most controllers for continuous time delay systems have (theoretically) infinite storage requirements.

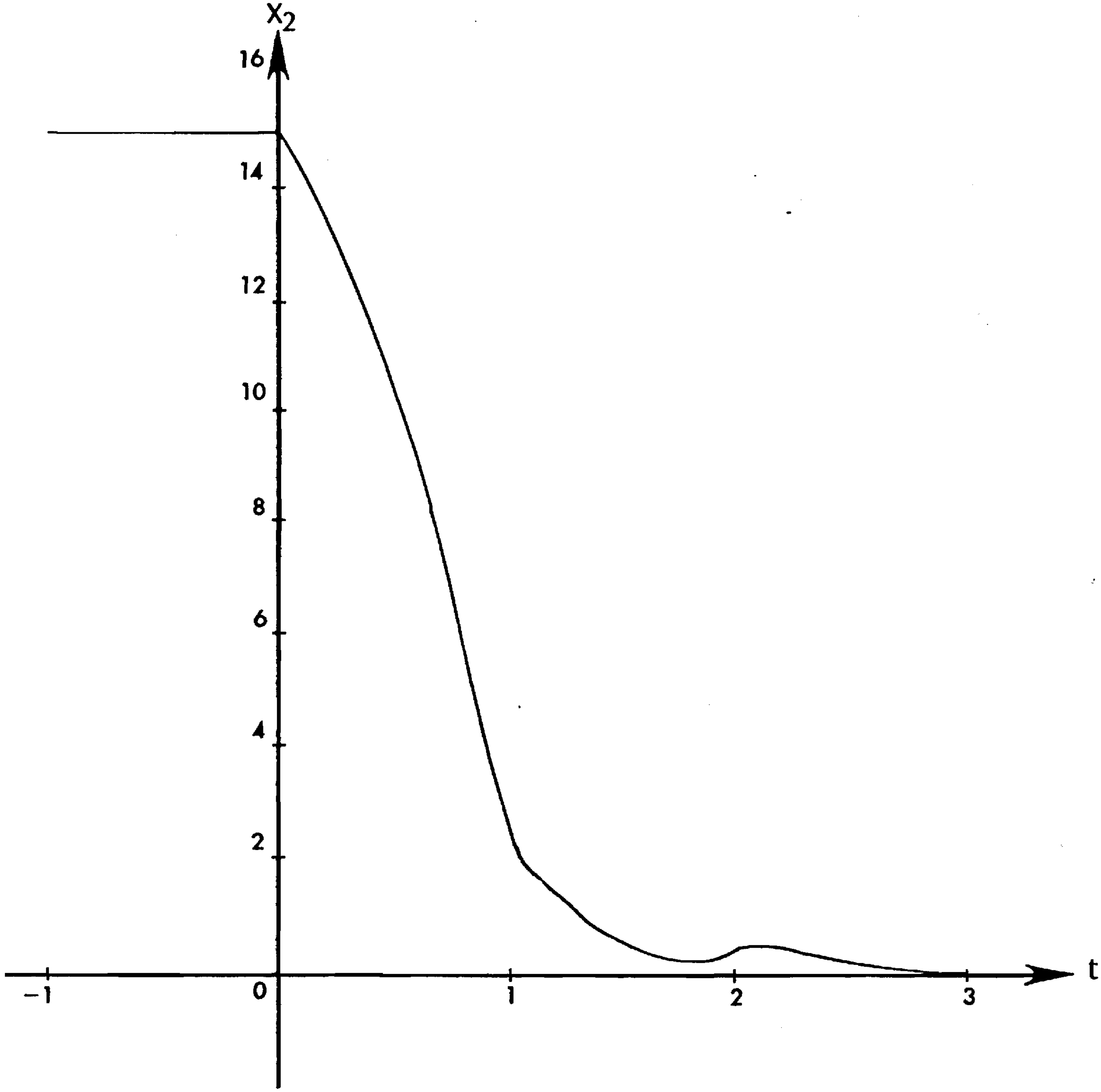
Figure Captions

Figure 1-a. Closed-loop trajectory for $x_1(t)$.

Figure 1-b. Closed-loop trajectory for $x_2(t)$.



X_1
10
8
6
4
2
-2
-1
0
1
2
3
t



X_2
16
14
12
10
8
6
4
2
0
−1
1
2
3
t

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ANALYSIS AND CONTROL OF TIME-DELAY SYSTEMS USING FINITE DIMENSIONAL APPROXIMATIONS

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Abstract

Analysis and control of time-delay systems governed by differential-difference equations are considered using finite dimensional approximations obtained from truncated Taylor series expansion of the delay terms. Through the use of simple examples a tutorial presentation is given of some of the problems in control and analysis caused by the finite dimensional models.

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ANALYSIS AND CONTROL OF TIME-DELAY SYSTEMS USING FINITE DIMENSIONAL APPROXIMATIONS

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1. Introduction

Infinite dimensional system problems are inherently more difficult to solve than similar problems for finite dimensional systems. Frequently, finite dimensional approximations are therefore used for analysis of infinite dimensional systems, and such approximations are also used to determine control inputs in the hope that the same inputs will also satisfactorily control the infinite dimensional system. Some of the pitfalls implicit in this approach when applied to analysis and control of time-delay systems are explained in this paper and illustrated by scalar examples. For simplicity only linear, constant-coefficient systems with a single delay are considered. In Section 2 the qualitative analysis of time-delay systems is considered and Section 3 deals with their control.

2. Qualitative Analysis

By making finite dimensional approximations of time-delay systems one may introduce qualitative properties distinctly different from those of the original delay system. This is shown to be the case in the following example [1, p. 254] when the approximation is obtained by truncating a Taylor series.

Example 1.

Consider the scalar system

$$\dot{x}(t) = -2x(t) + x(t-r) \tag{1}$$

with $r > 0$. We will show that all solutions to this equation decrease exponentially to zero as $t \to \infty$. To prove this, define the characteristic function

$$\Delta(\lambda) = \lambda + 2 - e^{-\lambda r} \tag{2}$$

and note that $\Delta(-2) < 0$ and $\Delta(0) > 0$. Viewed as a continuous function of a real variable, $\Delta(\lambda)$ must then have a zero between -2 and 0; i.e., $\Delta(-\hat{\lambda}) = 0$ for $\hat{\lambda} \varepsilon (0,2)$. Then

$$z(t) \triangleq x(t)\ e^{\hat{\lambda} t}$$

satisfies

$$\begin{aligned}\dot{z}(t) &= \hat{\lambda} z(t) + [-2x(t) + x(t-r)]e^{\hat{\lambda} t} \\ &= (\hat{\lambda}-2)\ [z(t) + z(t-r)]\end{aligned}$$

Define

$$v(t) \triangleq z^2(t) + (2-\hat{\lambda}) \int_{t-r}^{t} z^2(\tau)\ d\tau$$

and note that $\dot{v}(t) = -(2-\hat{\lambda})\ [z(t) - z(t-r)]^2 \le 0$. Since $v(t)$ is nondecreasing and $|z(t)|^2 \le v(t)$, $z(t)$ is bounded and

$$x(t) = z(t)\ e^{-\hat{\lambda} t} \to 0 \text{ as } t \to \infty$$

for every initial function.

On the other hand, consider the differential equation obtained by approximating $x(t-r)$ in (1) with the truncated Taylor series

$$x(t) - r\dot{x}(t) + \frac{1}{2} r^2\ \ddot{x}(t) - \ldots. + \frac{1}{(2k)!}\ r^{2k} x^{(2k)}(t),\ k \ge 1.$$

It is then easily seen that the resulting finite dimensional approximation of (1) has at least one eigenvalue with a positive real part so the general solution will increase exponentially with time. This contrasting asymptotic behavior exists for <u>all</u> even ordered truncations (i.e. irrespective of how large k is chosen).

3. Control

Standard methods for synthesizing feedback controls yield stable closed-loop systems. When these methods are applied to finite dimensional approximations of time-delay systems, the resulting closed-loop delay system may, however, be unstable. This is illustrated by example 2 where optimal control design is used to determine an asymptotically stable control for the approximate model.

Example 2.

Consider the system

$$\dot{x}(t) = 6x(t) - 7x(t-1) + u(t) \tag{3}$$

with the delay normalized to unity. Approximate the delay term by a truncated Taylor series so

$$x(t-1) \approx \dot{x}(t) - x(t).$$

For the resulting finite dimensional approximation

$$\dot{x}(t) = \frac{1}{6}\,x(t) - \frac{1}{6}\,u(t)$$

determine the unique optimal control u which minimizes the cost

$$\int_0^\infty [5x^2(t) + u^2(t)]\,dt$$

Then

$$u(t) = (1 + \sqrt{6}\,)\,x(t)$$

which substituted into (3) gives a closed-loop infinite dimensional system described by

$$\dot{x}(t) = (7 + \sqrt{6}\,)\,x(t) - 7x(t-1) \tag{5}$$

The characteristic equation for (5) can be written as

$$(\lambda - 7 - \sqrt{6}\,)\, e^{\lambda} + 7 = 0 \tag{6}$$

To study the roots of this transcendental equation we employ the following result due to N. D. Hayes [2].

Theorem 1.

All roots of the equation $(\lambda + \alpha)\, e^{\lambda} + \beta = 0$, where α and β are real, have negative real parts if and only if

(i) $\alpha > -1$

(ii) $\alpha + \beta > 0$

(iii) $\beta < \xi \sin \xi - \alpha \cos \xi$ where ξ is the root of $\xi = -\alpha \tan \xi$, $0 < \xi < \pi$, if $\alpha \neq 0$ and $\xi = \pi/2$ if $\alpha = 0$.

As it is easily verified that (6) has no purely imaginary root and does not satisfy the conditions of the theorem, there is at least one root of (6) with a positive real part. Hence system (5) is unstable.

In view of the previous example, it would be quite useful to know when a stabilizing feedback control $u(t) = cx(t)$ for the finite dimensional approximation

$$\dot{x}(t) = \frac{a + b}{1 + b}\, x(t) + \frac{1}{1 + b}\, u(t) \tag{7}$$

obtained from

$$\dot{x}(t) = ax(t) + bx(t-1) + u(t) \tag{8}$$

with $x(t-1) \approx x(t) - \dot{x}(t)$ would also stabilize the time-delay system (8). The result is given below:

Theorem 2.

The closed-loop delay system (8) with feedback control $u = cx$ is asymptotically stable whenever u makes (7) asymptotically stable and $b > -1$.

Proof.

With $b > -1$ asymptotic stability of (7) implies that the first two conditions of theorem 1 are satisfied. The third condition can be rewritten as $b > -\frac{d}{\cos \xi}$ with $d \triangleq c + a$. Since $\cos \xi = d \sin \xi/\xi$, the inequality be- becomes $b > -\frac{\xi}{\sin \xi}$. The function $f(\xi) \triangleq \frac{\xi}{\sin \xi}$ satisfies $f'(\xi) > 0$ on $(0,\pi)$ and $\lim_{\xi \to 0} f(\xi) = 1$. Hence condition (iii) of theorem 1 is trivially satisfied.

Retaining more terms in the truncated Taylor series expansion of the delayed term will often make the behaviors of the finite dimensional and the infinite dimensional feedback systems more similar. It still is surprising that with a second order approximation, as shown below, asymptotic stability of the finite dimensional approximation guarantees asymptotic stability of the closed-loop delay system. With truncated expansion

$$x(t - 1) \approx x(t) - \dot{x}(t) + \frac{1}{2} \ddot{x}(t) \qquad (9)$$

and feedback control $u(t) = cx(t)$ the finite dimensional approximation becomes the second order system

$$\ddot{x}(t) - 2(1 + \frac{1}{b}) \dot{x}(t) + 2(1 + \frac{a + c}{b}) x(t) = 0 \qquad (10)$$

This system is asymptotically stable if and only if $1 + \frac{1}{b} < 0$ and $1 + \frac{a + c}{b} > 0$. These conditions imply that conditions (i) and (ii) for asymptotic stability of

$$\dot{x}(t) = (a + c) x(t) + bx(t - 1) \qquad (11)$$

are satisfied. Since $b > -1$, theorem 2 insures that (11) is asymptotically stable.

It is interesting to note that Driver, Sasser, and Slater [3] have shown that under certain conditions the asymptotic behavior of solutions of the undriven systems

$(t-\tau)$

$$\dot{x}(t) = ax(t) + bx(t)$$

and

$$\dot{x}(t) = ax(t) + bx(t) - b\tau x(t)$$

is qualitatively the same.

It is a simple matter to extend our result to the set of controls which are linear combinations of $x(t)$ and $\dot{x}(t)$. With $u = cx + d\dot{x}$ the proof is identical to that above with $\hat{a} = a(1 - d)^{-1}$, $\hat{b} = b(1 - d)^{-1}$, and $\hat{c} = (1 - d)^{-1}$ replacing a, b, and c. We then have:

Theorem 3.

The closed-loop time-delay system (8) with control $u(t) = cx(t) + d\dot{x}(t)$ is asymptotically stable whenever u makes the approximate model obtained by (9) asymptotically stable.

4. Conclusions

The examples above show that care must be exercised when using finite dimensional approximations for analysis and control of time-delay systems. However, under certain conditions any control which makes the approximate model asymptotically stable will also insure asymptotic stability of the closed-loop time-delay system. The main obstacle to establishing such conditions beyond the scalar case is the problem of determining the roots of transcendental equations.

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MEMORYLESS STABILIZATION OF LINEAR DELAY-DIFFERENTIAL SYSTEMS

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Abstract

Sufficient conditions for memoryless feedback stabilization of linear time-invariant delay-differential systems are derived for both constant and time-dependent delays.

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MEMORYLESS STABILIZATION OF LINEAR DELAY-DIFFERENTIAL SYSTEMS

1. Introduction

Stabilization of the delay-differential system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + Cu(t) \tag{1}$$

with $x(t)\epsilon R^n$, $u(t)\epsilon R^p$, $\tau > 0$, has been considered by several researchers, see e.g. [1] - [6]. Common to their work is the use of the state $x_t \triangleq x(t+\theta)$, $\theta\epsilon[-\tau,0]$ or other sets of past values of $x\epsilon R^n$ to stabilize the system. In this note memoryless stabilization [7,8], for which the stabilizing linear feedback controller $u(t) = Lx(t)$ at time t requires only $x(t)$ as input, is considered. It will be shown by an example that a previous theorem [7] is incorrect and a corrected version of this theorem will be given. Furthermore, the theorem is extended to cover memoryless feedback stabilization of linear systems with variable delay.

2. An Example

Consider the scalar system

$$\dot{x}(t) = ax(t) + bx(t-\tau) + u(t) \tag{2}$$

with $a < 0$, $\tau > 0$, $b = a\coth(aT)$, $T > 0$. Clearly the system described by (2) is both controllable and pointwise complete. According to [7] the feedback system obtained from (2) with $u(t) = -W^{-1}x(t)$ is asymptotically stable when $Q = 1 + e^{-2aT}$, $W = (1-e^{-2aT})/2a$ (In this case $Q^2 - 4W^2b^2 = 0$, cfr. proof in [7]). However, the feedback system is

$$\dot{x}(t) = - a \coth(aT)\, x(t) + bx(t-\tau) \tag{3}$$

with characteristic equation

$$s + a \coth(aT) - a \coth (aT)e^{-s\tau} = 0$$

which has a root at $s=0$. Consequently, system (3) is not asymptotically stable. The class of counterexamples also includes the system where $a=0$ in which case the coefficient $b=a\coth(aT)$ in (2) is replaced by its limiting value as $a\to 0$, namely, $b = \frac{1}{T}$.

3. Revised Theorem.

Retaining some of the features of the theorem in [7], we first prove the following result.

Theorem 1.

If (A,C) is controllable and the condition

$$CC^T + e^{-AT}CC^Te^{-A^TT} - Q - WB^TQ^{-1}BW > 0$$

where $W = \int_0^T \exp(-As)CC^T\exp(-A^Ts)ds$, $T > 0$ is satisfied for some $Q > 0$, then (1) is stabilized by the control law $u(t) = -C^TW^{-1}x(t)$.

Proof

Consider the system

$$\dot{z}(t) = (A - CC^TW^{-1})^Tz(t) + B^Tz(t-\tau) \tag{4}$$

and the Liapunov functional

$$V(z_t) \triangleq z^T(t)Wz(t) + \int_{t-\tau}^{t} z^T(s)Qz(s)ds$$

(Note that $V(z_t) > 0$ whenever $z_t \neq 0$ by the assumptions on (A,C) and Q). Then, along a trajectory of (4),

$$\dot{V}(z_t) = - \begin{bmatrix} z(t) \\ z(t-\tau) \end{bmatrix}^T \begin{bmatrix} CC^T + e^{-AT}CC^T e^{-A^TT} - Q & -WB^T \\ -BW & Q \end{bmatrix} \begin{bmatrix} z(t) \\ z(t-\tau) \end{bmatrix}$$

since $AW + WA^T = CC^T - e^{-AT}CC^Te^{-A^TT}$ [11]. There exists a constant $\beta > 0$ such that $\dot{V}(z_t) \leq -\beta\| z_t \|$ provided (1) $Q > 0$ and (2) $CC^T + e^{-AT}CC^Te^{-A^TT} - Q - WB^TQ^{-1} BW > 0$ [9]. From comparison of the characteristic equations it follows that the system

$$\dot{x}(t) = (A - CC^TW^{-1})\ x(t) + Bx(t-\tau) \tag{5}$$

is asymptotically stable. This completes the proof.

The stabilization procedure works for any positive T and is also independent of the delay τ. This independence will be used to extend the theorem to systems with time-dependent delays.

4. Extension.

Consider the problem of memoryless feedback stabilization of the variable delay system

$$\dot{x}(t) = Ax(t) + Bx(t-h(t)) + Cu(t), \quad t > 0 \tag{6}$$

where $0 \leq h(t)$ is a continuously differentiable function of time satisfying $h'(t) < 1$ for all $t > 0$. By defining

$$V(z_t) \triangleq z^T(t)\, Wz(t) + \int_{t-h(t)}^{t} z^T(s)\, Qz(s)\, ds$$

we obtain along a trajectory of the transpose system

$$\dot{z}(t) = (A - CC^T W^{-1})^T z(t) + B^T z(t-h(t)) ,$$

$$\dot{V}(z_t) = -\begin{bmatrix} z(t) \\ z(t-h(t)) \end{bmatrix}^T \begin{bmatrix} CC^T + e^{-AT}CC^T e^{-A^T T} - Q & -WB^T \\ -BW & (1-h'(t))Q \end{bmatrix} \begin{bmatrix} z(t) \\ z(t-h(t)) \end{bmatrix}$$

Hence, we have:

Theorem 2.

If (A,C) is controllable and the condition $CC^T + e^{-AT}CC^T e^{-A^T T} - Q - WB^T Q^{-1} BW(1-h'(t))^{-1} > 0$ where $W = \int_0^T \exp(-As)CC^T \exp(-A^T s)ds$, $\tau > 0$ is satisfied for some $Q > 0$, then (6) is stabilized by the control law $u(t) = -C^T W^{-1} x(t)$.

5. Conclusion.

Sufficient conditions for memoryless stabilization of linear time-invariant delay systems were obtained. The stabilizing feedback gain is determined algebraically from the system matrices instead of from a matrix differential equation as in [8]. The extension of the result to systems with time-dependent delays complements the work of Ikeda and Ashida [10] on memoryless stabilization of delay systems with time-dependent parameters.

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Part II

1. Summary

The research carried out by Professor Kamen under NSF Grant No. ENG78-12231, which provided a total of two-man months of support for Professor Kamen, dealt with a new approach to stability and stabilization of linear systems with time delays. In particular, the work centered on the development of constructive techniques for determining stability and stabilizability independent of delay with a given order. A complete description of the results is contained in the attached paper "Linear systems with commensurate time delays: Stability and stabilization independent of delay." This paper has been submitted to the IEEE Transactions on Automatic Control for publication.

LINEAR SYSTEMS WITH COMMENSURATE TIME DELAYS:

STABILITY AND STABILIZATION INDEPENDENT OF DELAY*

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ABSTRACT

Notions of exponential stability independent of delay and stabilizability independent of delay are developed for the class of delay differential systems of the retarded type with commensurate time delays. Various criteria for exponential stability independent of delay with a given order are specified in terms of matrices whose entries are functions of a single real parameter and polynomials in one variable whose coefficients are functions of a single real parameter. Sufficient conditions and a necessary condition based on local stabilizability are given for stabilizability independent of delay using state feedback with commensurate time delays. Constructive methods for determining a stabilizing feedback are also presented. The last part of the paper deals with a standard type of observer and regulator with the requirement that the closed-loop system be stable independent of delay.

I. INTRODUCTION

Given a delay differential system (or equation) of the retarded type with delays equal to integer multiples of a fixed delay $h > 0$, it is well known [1],[2] that the system is asymptotically stable if and only if the characteristic function $P(s,e^{-hs})$ satisfies the condition

$$P(s,e^{-hs}) \neq 0, \ \mathrm{Re}\, s \geqq 0. \tag{1.1}$$

The characteristic function $P(s,e^{-hs})$ is a polynomial in s and e^{-hs}, sometimes called an exponential polynomial or a quasipolynomial.

Although there are several methods for determining whether or not (1.1) holds (see [1, Chapter 13] and [3, Chapter 3]), the computational aspects of these methods are very complex in general. One way to avoid this difficulty is to consider a stronger notion of stability which can be tested for using standard techniques. One such notion is asymptotic stability <u>independent of delay</u>, which is characterized by the condition

$$P(s,e^{-as}) \neq 0, \ \mathrm{Re}\, s \geqq 0, \ \text{all real numbers } a \geqq 0. \tag{1.2}$$

In [4] it is shown that (1.2) is equivalent to

$$P(s,e^{i\omega}) \neq 0, \ \mathrm{Re}\, s \geqq 0, \ \omega \in [0,2\pi], \text{ where } i = \sqrt{-1}. \tag{1.3}$$

For any fixed value of $\omega \in [0,2\pi]$, $P(s,e^{i\omega})$ is a polynomial in s with (in general) complex coefficients. Hence one can test for asymptotic stability independent of delay by applying on a point-by-point basis existing stability tests for ordinary polynomials. (Details are given in Section III.)

Asymptotic stability independent of delay (i.o.d.) is obviously a much stronger property than asymptotic stability; but the set of all polynomials in s and e^{-as} satisfying (1.2) is large enough to be interesting. In this paper we develop the notion of exponential stability i.o.d., and then apply this concept to the study of feedback control by requiring that the resulting closed-loop system be stable i.o.d.. The approach presented here fits in very nicely with the algebraic theory of linear systems defined over rings or algebras, as applied to systems with time delays (e.g., see [5]-[20]). Part of this previous work deals with the problem of pole assignment. For systems with time delays, pole assignability using state feedback or dynamic output feedback implies that we can construct a closed-loop system with characteristic function $P_{cl}(s,e^{-hs}) = P(s)$ = a polynomial in s with real coefficients, and with the zeros of P(s) arbitrarily assigned (up to complex conjugate pairs). A key point here is that since we can get a closed-loop characteristic function that is independent of e^{-hs}, pole assignability implies that we can construct a closed-loop system that is stable independent of delay. In general, stabilizability independent of delay is a much weaker property than pole assignability. This observation provided part of the original motivation for considering the notion of stability independent of delay.

After the preliminaries given in the next section, in Section III we define the notion of γ-stability independent of delay, and then present various criteria for this property. In Sections IV-VI, we study the notion of γ-stabilizability independent of delay using polynomial feedback. Sufficient conditions for this property are given in Section IV, while a necessary condition based on the notion of local γ-stabilizability is given in Section V. Constructive techniques for determining a stabilizing feedback are considered in Section VI. Finally, a brief sketch of a standard type of observer and regulator is presented in Section VII.

II. EXPONENTIAL STABILITY

In this section we give a precise definition of exponential stability with a given order for linear systems with commensurate time delays. To do this, we need to consider a space of function segments with an appropriate norm. Any reader who would like to bypass the following technicalities may go directly to Section III.

Let $\mathbb{R}$ denote the field of real numbers, and for any positive integer n, let $\mathbb{R}^n$ denote the space of n-element column vectors over $\mathbb{R}$. Given $a \in \mathbb{R}$ with $a > 0$, let $\mathscr{C}([-a,0];\mathbb{R}^n)$ denote the space of continuous functions defined on $[-a,0]$ with values in $\mathbb{R}^n$. Given $\varphi \in \mathscr{C}([-a,0];\mathbb{R}^n)$, we define the norm $\|\varphi\|$ by

$$\|\varphi\| = \sup_{\theta \in [-a,0]} \|\varphi(\theta)\| \;, \tag{2.1}$$

where $\|\varphi(\theta)\|$ denotes the Euclidean norm of $\varphi(\theta) \in \mathbb{R}^n$. With the norm (2.1), $\mathscr{C}([-a,0];\mathbb{R}^n)$ is a Banach space.

We shall study the class of linear systems with commensurate time delays given by the dynamical equations

$$\begin{aligned} \frac{dx(t)}{dt} &= \sum_{k=0}^{q} F_k x(t-kh) + \sum_{k=0}^{r} G_k u(t-kh), \quad t>0, \\ y(t) &= \sum_{k=0}^{\mu} H_k x(t-kh), \end{aligned} \tag{2.2}$$

where $h > 0$ is a fixed delay and the F_k (resp., the G_k, H_k) are $n \times n$ (resp., $n \times m$, $p \times n$) matrices over $\mathbb{R}$. In (2.2), $x(t) \in \mathbb{R}^n$ is the "instantaneous state" at time t, $u(t) \in \mathbb{R}^n$ is the input or control at time t, and $y(t) \in \mathbb{R}^p$ is the output at time t. The complete state at time t of the system (2.2) is the function segment x_t defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-qh,0]$. The initial function x_0 of the system (2.2) will be assumed to be an element of $\mathscr{C}([-qh,0];\mathbb{R}^n)$.

Now consider the free or unforced behavior of the system given by

$$\frac{dx(t)}{dt} = \sum_{k=0}^{q} F_k x(t-kh), \quad t>0, \tag{2.3}$$

with initial function $x_0 = \varphi \in \mathscr{C}([-qh,0];\mathbb{R}^n)$. It should be noted that we could consider initial functions φ which are not continuous, but we will not do so here (see [21]). It is known [2] that for any initial function $\varphi \in \mathscr{C}([-qh,0];\mathbb{R}^n)$, (2.3) has a unique solution $x(t)$ with $x_t \in \mathscr{C}([-qh,0];\mathbb{R}^n)$ for $t>0$.

Definition 1: Let β be a fixed nonzero positive number. The system (2.2) is said to be exponentially stable with order β if there exists a constant K such that for any $\varphi \in \mathscr{C}([-qh,0];\mathbb{R}^n)$,

$$\sup_{\theta \in [-qh,0]} \| x(t+\theta) \| = \| x_t \| \leqq Ke^{-\beta t} \| \varphi \|, \ t>0,$$

where $x(t)$ is the solution to (2.3) with initial function φ.

Now define $F(e^{-hs}) = \sum_{k=0}^{q} F_k e^{-khs}$, and let $P(s,e^{-hs})$ denote the characteristic function associated with (2.3); that is,

$$P(s,e^{-hs}) = \det(sI - F(e^{-hs})),$$

where det denotes the determinant and I is the $n \times n$ identity matrix.

We then have the following condition for stability, the proof of which follows directly from the results in Hale's book [2].

Theorem 1: Given a fixed real number $\gamma \geqq 0$, suppose that

$$P(s, e^{-hs}) \neq 0, \ \mathrm{Re}\, s \geqq -\gamma.$$

Then the system (2.2) is exponentially stable with order β for some $\beta > \gamma$.

III. STABILITY INDEPENDENT OF DELAY

Given the system (2.2), define

$$F(z) = \sum_{k=0}^{q} F_k z^k, \ G(z) = \sum_{k=0}^{r} G_k z^k, \ H(z) = \sum_{k=0}^{\mu} H_k z^k,$$

where z will usually be viewed as a complex variable. In the remainder of the paper, we will often denote the system (2.2) by the triple $(F(z),G(z),H(z))$.

Let $P(s,z) = \det(sI - F(z)$, so that the characteristic function of the system $(F(z),G(z),H(z))$ is equal to $P(s,z)$ with $z = e^{-hs}$. For any real number a, let $P(s,e^{-as})$ denote $P(s,z)$ with $z = e^{-as}$.

Definition 2: Given a fixed nonnegative real number γ, the system $(F(z),G(z),H(z))$ is γ-stable independent of delay (i.o.d.) if and only if

$$P(s,e^{-as}) \neq 0, \ \mathrm{Re}\, s \geqq -\gamma, \ \text{all real numbers } a \geqq 0. \tag{3.1}$$

By Theorem 1, γ-stability i.o.d. implies that the system is exponentially stable independent of delay with order $> \gamma$. More precisely, if for any $a \geqq 0$ we set $h = a$ in (2.2), the system (2.2) is exponentially stable with order $\beta_a > \gamma$, where β_a may depend on a.

Now for any fixed complex number z, let $\lambda_j(F(z))$, $j = 1,2,\ldots,n$, denote the eigenvalues of $F(z)$, and for any $\omega \in [0,2\pi]$, let $P(s,e^{h\gamma + i\omega})$ denote

$P(s,z)$ with $z = e^{h\gamma + i\omega}$. We then have the following characterizations of γ-stability i.o.d..

Theorem 2: For any fixed $\gamma \geqq 0$, the following conditions are equivalent.

(1) The system $(F(z),G(z),H(z))$ is γ-stable i.o.d.;

(2) $P(s,z) \neq 0$, $\mathrm{Re}\, s \geqq -\gamma$, $|z| \leqq e^{h\gamma}$;

(3) $P(s,e^{h\gamma + i\omega}) \neq 0$, $\mathrm{Re}\, s \geqq -\gamma$, $\omega \in [0,2\pi]$;

(4) $\mathrm{Re}\,\lambda_j(F(z)) < -\gamma$, $j = 1,2,\ldots,n$, $|z| \leqq e^{h\gamma}$;

(5) $\mathrm{Re}\,\lambda_j(F(e^{h\gamma + i\omega})) < -\gamma$, $j = 1,2,\ldots,n$, $\omega \in [0,2\pi]$.

Proof: For $\gamma = 0$, the equivalence between (1),(2), and (3) is established in [4, Theorem 2]. Via a simple change of variables, it follows that (1),(2), and (3) are equivalent for any $\gamma > 0$. The equivalence between (2) and (4) and the equivalence between (3) and (5) follow from the relationship $P(s,z) = \det(sI - F(z))$.

As seen from the following result, in testing for conditions (3) and (5) in Theorem 2, one only needs to take $\omega \in [0,\pi]$.

Corollary: For any fixed $\gamma \geqq 0$, the following conditions are equivalent.

(1) The system $(F(z),G(z),H(z))$ is γ-stable i.o.d.;

(2) $P(s,e^{h\gamma + i\omega}) \neq 0$, $\mathrm{Re}\, s \geqq -\gamma$, $\omega \in [0,\pi]$;

(3) $\mathrm{Re}\,\lambda_j(F(e^{h\gamma + i\omega})) < -\gamma$, $j = 1,2,\ldots,n$, $\omega \in [0,\pi]$.

Proof: Since the coefficients of the matrix polynomial $F(z)$ are over the reals, the complex conjugate $\overline{F(e^{h\gamma + i\omega})}$ is equal to $F(e^{h\gamma - i\omega})$. Thus if λ is an eigenvalue of $F(e^{h\gamma + i\omega})$, the complex conjugate $\bar{\lambda}$ is an eigenvalue of $F(e^{h\lambda - i\omega})$. In addition, $F(e^{h\gamma + i\omega})$ is a periodic matrix function of ω with period 2π, and hence $\mathrm{Re}\,\lambda_j(F(e^{h\gamma + i\omega})) < -\gamma$, all $\omega \in [0,2\pi]$ if and only if $\mathrm{Re}\,\lambda_j(F(e^{h\lambda + i\omega})) < -\gamma$, all $\omega \in [0,\pi]$.

For any fixed $\omega \in [0,\pi]$, $P(s,e^{h\gamma + i\omega})$ is a polynomial in s with (in general) complex coefficients and $F(e^{h\gamma + i\omega})$ is a matrix with complex entries. Hence one can test for conditions (2) and (3) in the corollary by applying point-by-point existing stability tests for ordinary polynomials and matrices. Note that in contrast, for a fixed value of $a > 0$, $P(s,e^{-as})$ is not in general a polynomial in s with real or complex coefficients. Thus it is somewhat surprising that γ-stability i.o.d. is equivalent to the conditions in the corollary.

Since condition (3) in the corollary is specified completely in terms of the coefficient matrix $F(z)$ of the system, we can test for γ-stability i.o.d. without having to compute the characteristic function $P(s,e^{-hs})$. This capability of circumventing the computation of $P(s,e^{-hs})$ may be highly desirable, because as the size of the matrix $F(z)$ increases, computational errors in the calculation of $P(s,e^{-hs}) = \det(sI - F(e^{-hs}))$ can lead to serious problems in the accuracy of any stability test based on $P(s,e^{-hs})$.

In testing for condition (2) or (3) in the corollary, one can use the property that the eigenvalues of $F(e^{h\gamma + i\omega})$ or the zeros of $P(s,e^{h\gamma + i\omega})$ are continuous functions of ω. In particular, if for some $\omega_0 \in [0,\pi]$, $\operatorname{Re} \lambda_j(F(e^{h\gamma + i\omega_0})) > -\gamma$ for some j or $P(s,e^{h\gamma + i\omega_0}) = 0$ for some s with $\operatorname{Re} s > -\gamma$, by continuity there must exist an open neighborhood $V_0 \subset \mathbb{R}$ of the point ω_0 such that for every $\omega \in V_0$, $\operatorname{Re} \lambda_j(F(e^{h\gamma + i\omega})) > -\gamma$ or $P(s,e^{h\gamma + i\omega}) = 0$ for some s (depending on ω) with $\operatorname{Re} s > -\gamma$. Therefore, assuming that ω_0 is not known a priori, by testing (2) or (3) at a suitable number of evenly-spaced grid points $0 = \omega_1 < \omega_2 < \cdots < \omega_N = \pi$, one would discover that the system is not γ-stable i.o.d.. Of course, a problem with this approach is determining the number N of points in the grid. The usual procedure is to keep refining the grid until a negative result is established or until it's "clear" that the process is converging to the conclusion that condition (2) or (3) is satisfied.

An obvious consequence of the continuity property mentioned above is that the eigenvalues of $F(e^{h\gamma + i\omega})$ or the zeros of $P(s,e^{h\gamma + i\omega})$ can be computed using iterative techniques: The eigenvalues of $F(e^{h\gamma + i\omega})$ or the zeros of $P(s,e^{h\gamma + i\omega})$ at $\omega = \omega_k$ can be used as an initial estimate of the eigenvalues or zeros at $\omega = \omega_{k+1} > \omega_k$. Iterative techniques for computing eigenvalues of matrices or the zeros of polynomials are described in [22].

In testing for condition (2) or (3) on a point-by-point basis, it is of course not necessary to compute the eigenvalues of $F(e^{h\gamma + i\omega})$ or the zeros of $P(s,e^{h\gamma + i\omega})$. For instance, we can test for condition (3) by applying the Lyapunov stability criterion for matrices over $\mathbb{R}$ or $\mathbb{C}$ (= field of complex numbers), and we can test for condition (2) by applying the Hermite matrix criterion. The latter procedure is described below.

For any fixed $\gamma \geqq 0$, let $P_\gamma(s,z) = \det((s-\gamma)I - F(e^{h\gamma}z))$. It follows that condition (2) in the corollary is equivalent to

$$P_\gamma(s,e^{i\omega}) \neq 0, \ \operatorname{Re} s \geqq 0, \ \omega \in [0,\pi].$$

Now by definition of $P_\gamma(s,z)$, we have that

$$P_\gamma(s,e^{i\omega}) = s^n + \sum_{k=1}^{n} b_k(e^{i\omega})s^{n-k},$$

where the $b_k(e^{i\omega})$ are polynomials in $e^{i\omega}$ with real coefficients. For each $\omega \in [0,\pi]$, let $H_\gamma(\omega)$ denote the $n \times n$ Hermite matrix [23, p. 80] associated with $P_\gamma(s,e^{i\omega})$, viewed as a polynomial in s with complex coefficients. The $\mu\nu$ entry $h_{\mu\nu}$ of $H_\gamma(\omega)$ is given by

$$h_{\mu\nu} = \sum_{k=1}^{\mu} i^{\mu+\nu}\left[(-1)^{\nu+k-1}\, b_{\mu-k}\bar{b}_{\nu+k-1} + (-1)^{\mu-k+1}\,\bar{b}_{\mu-k}b_{\nu+k-1}\right], \tag{3.2}$$

where superscript "bar" denotes the complex conjugate. In evaluating (3.2), we need to take $b_0 = 1$ and $b_k = 0$ for $k > n$. It follows directly from (3.2) that $\bar{h}_{\mu\nu} = h_{\mu\nu} = h_{\nu\mu}$. Hence $H_\gamma(\omega)$ is a real symmetric matrix for every $\omega \in [0,\pi]$.

Theorem 3: For any fixed $\gamma \geqq 0$, the following conditions are equivalent.

(1) The system $(F(z),G(z),H(z))$ is γ-stable i.o.d.;

(2) $H_\gamma(\omega)$ is positive definite for all $\omega \in [0,\pi]$;

(3) $H_\gamma(0)$ is positive definite and $\operatorname{rank} H_\gamma(\omega) = n$ for all $\omega \in [0,\pi]$;

(4) $H_\gamma(0)$ is positive definite and $\det H_\gamma(\omega) \neq 0$ for all $\omega \in [0,\pi]$.

Proof: The equivalence between (1) and (2) follows from Theorem 2 by applying the Hermite stability criterion point-by-point. The equivalence between (2) and (3) follows from the property that the principal minors of $H_\gamma(\omega)$ are continuous functions of ω. Finally, (3) $\Leftrightarrow$ (4) is obvious.

It follows directly from Siljak's constructions [24] that one can determine whether or not condition (4) in Theorem 3 holds in a finite number of steps. In other words, there is a finite test for γ-stability i.o.d.. This was first pointed out in [4] for the special case $\gamma = 0$.

We should mention that results on stability independent of delay are derived in [25] for systems with delays only in the off-diagonal interactions; that is, systems given by

$$\frac{dx_i(t)}{dt} = a_{ii}x_i(t) + \sum_{j\neq i} a_{ij}x_j(t - T_{ij}),\ i = 1,2,\ldots,n.$$

The approach taken in [25] is quite different from that considered here.

IV. STABILIZABILITY INDEPENDENT OF DELAY

Given the system $(F(z),G(z),H(z))$ defined by (2.2), let's now consider state feedback by setting

$$u(t) = -\sum_{k=0}^{\eta} L_k x(t-kh),$$

where the L_k are $m \times n$ matrices over $\mathbb{R}$. The state equation for the resulting closed-loop system is given by

$$\frac{dx(t)}{dt} = \sum_{k=0}^{q} F_k x(t-kh) - \sum_{k=0}^{r} \sum_{j=0}^{\eta} G_k L_j x(t-kh-jh). \tag{4.1}$$

Defining $L(z) = \sum_{k=0}^{\eta} L_k z^k$, we have the following expression for the characteristic function $P_{cl}(s,e^{-hs})$ of the closed-loop system:

$$P_{cl}(s,e^{-hs}) = \det(sI - F(z) + G(z)L(z))\Big|_{z=e^{-hs}} .$$

Definition 3: Given a fixed real number $\gamma \geqq 0$, the system $(F(z),G(z),H(z))$ is γ-stabilizable independent of delay if and only if there exists a polynomial feedback matrix $L(z)$ such that the closed-loop system (4.1) is γ-stable i.o.d..

We shall first give sufficient conditions for γ-stabilizability i.o.d.. In the following development, $U(z)$ will denote the $n \times mn$ polynomial matrix defined by

$$U(z) = [G(z) \quad F(z)G(z) \quad \cdots \quad F^{n-1}(z)G(z)].$$

Theorem 4: Suppose that

$$\operatorname{rank} U(z) = n \quad \text{for all } z \in \mathbb{C}. \tag{4.2}$$

Then the system $(F(z),G(z),H(z))$ is γ-stabilizable i.o.d. for any $\gamma \geqq 0$.

Theorem 4 is actually a corollary of Morse's Theorem [6]: Suppose that the rank condition (4.2) holds. Then by Sontag's local criterion for reachability [7, p. 19], the columns of $U(z)$ generate the module of all n-element column vectors over the ring $\mathbb{R}[z]$ of polynomials in z with real coefficients. Hence by Morse's Theorem, for any self-conjugate set $\{e_1, e_2, \ldots, e_n\}$ of n complex numbers, there is a polynomial feedback matrix $L(z)$ such that

$$\det(sI - F(z) + G(z)L(z)) = \prod_{k=1}^{n} (s - e_k).$$

Therefore, given any $\gamma \geqq 0$, if we choose the e_k so that $\operatorname{Re} e_k < -\gamma$ for all k, the closed-loop system is γ-stable i.o.d..

It turns out that there is a weaker version of the rank condition (4.2) which still implies γ-stabilizability i.o.d..

Theorem 5: Given a fixed $\gamma \geqq 0$, suppose that

$$\operatorname{rank} U(z) = n, \ |z| \leqq e^{h\gamma}. \tag{4.3}$$

Then the system $(F(z),G(z),H(z))$ is γ-stabilizable i.o.d..

The proof of Theorem 5 is based on a normed algebra of rational functions in z defined as follows. Let $\mathbb{R}_\gamma(z)$ denote the set of all rational functions $\frac{a(z)}{b(z)}$, where $a(z)$ and $b(z)$ are polynomials in z with real coefficients and where $b(z) \neq 0$, $|z| \leqq e^{h\gamma}$. With the usual operations, $\mathbb{R}_\gamma(z)$ is an algebra; in fact, $\mathbb{R}_\gamma(z)$ is a principal ideal domain (The algebra $\mathbb{R}_0(z)$ appears in the work of Sontag [7, p. 24].) In addition, $\mathbb{R}_\gamma(z)$ is a normed algebra with the norm

$$\left\| \frac{a(z)}{b(z)} \right\| = \sup_{|z| \leqq e^{h\gamma}} \left| \frac{a(z)}{b(z)} \right| .$$

From known results [26, Theorem 16.6.4, p. 303], for any $\frac{a(z)}{b(z)} \in \mathbb{R}_\gamma(z)$ and any $\epsilon > 0$, there is a polynomial $c(z) \in \mathbb{R}[z]$ such that

$$\left\| \frac{a(z)}{b(z)} - c(z) \right\| < \epsilon.$$

The norm on $\mathbb{R}_\gamma(z)$ can be extended to $m \times n$ matrices $Q(z)$ over $\mathbb{R}_\gamma(z)$ as follows. With $q_{jk}(z)$ equal to the jk entry of $Q(z)$, define

$$\|Q\| = \sup_{|z| \leqq e^{h\gamma}} \|Q(z)\|, \text{ where } \|Q(z)\| = \max_j \sum_k |q_{jk}(z)|.$$

It follows that for any $m \times n$ matrix Q over $\mathbb{R}_\gamma(z)$ and any $\epsilon > 0$, there is a $m \times n$ matrix $\tilde{Q}$ over $\mathbb{R}[z]$ such that $\|Q - \tilde{Q}\| < \epsilon$.

Proof of Theorem 5: Suppose that the rank condition (4.3) is satisfied. Since $\mathbb{R}_\gamma(z)$ is a principal ideal domain, it follows from Morse's Theorem [6] that for any set $\{e_1, e_2, \ldots, e_n\}$ of n distinct real numbers with $e_k < -\gamma$ for all k, there is a $m \times n$ matrix $Q(z)$ over $\mathbb{R}_\gamma(z)$ such that

$$\det(sI - F(z) + G(z)Q(z)) = \prod_{k=1}^{n} (s - e_k). \tag{4.4}$$

Now for any $m \times n$ matrix $\tilde{Q}(z)$ over $\mathbb{R}[z]$, we have that

$$F(z) - G(z)\tilde{Q}(z) = F(z) - G(z)Q(z) + G(z)(Q(z) - \tilde{Q}(z)). \tag{4.5}$$

By (4.4),(4.5), and a known result [27, p. 234] on the perturbation of simple matrices over $\mathbb{C}$, for any fixed z with $|z| \leqq e^{h\gamma}$ the eigenvalues of $F(z) - G(z)\tilde{Q}(z)$

lie in at least one of the discs

$$|s - e_k| \leqq \| G(z)(Q(z) - \widetilde{Q}(z)) \| \nu(F(z) - G(z)Q(z)), k = 1,2,\dots,n, \tag{4.6}$$

where $\nu(F(z) - G(z)Q(z))$ is the condition number [27, p. 232] of $F(z) - G(z)Q(z)$. Since the eigenvalues of $F(z) - G(z)Q(z)$ are equal to $e_1, e_2, \dots, e_n$ for $|z| \leqq e^{h\gamma}$, where the e_k are distinct real numbers, it is not difficult to verify that

$$\sup_{|z| \leqq e^{h\gamma}} \nu(F(z) - G(z)Q(z)) = C < \infty.$$

Taking the supremum over $|z| \leqq e^{h\gamma}$ of the right side of (4.6), we have that for any z with $|z| \leqq e^{h\gamma}$ the eigenvalues of $F(z) - G(z)\widetilde{Q}(z)$ lie in at least one of the discs

$$|s - e_k| \leqq \| G(Q - \widetilde{Q}) \| C, k = 1,2,\dots,n. \tag{4.7}$$

Now since $\| G(Q - \widetilde{Q}) \| \leqq \| G \| \| Q - \widetilde{Q} \|$, by the above result on polynomial approximations, we can choose the polynomial matrix $\widetilde{Q}$ so that $\| G(Q - \widetilde{Q}) \|$ is as small as desired. Therefore, by (4.7) we can choose $\widetilde{Q}$ so that the real parts of the eigenvalues of $F(z) - G(z)\widetilde{Q}(z)$ are less than $-\gamma$ for $|z| \leqq e^{h\gamma}$. Hence by Theorem 2, the closed-loop system with $L(z) = \widetilde{Q}(z)$ is γ-stable i.o.d..

Note that by taking $\gamma = 0$ in Theorem 5, we have that $\operatorname{rank} U(z) = n$ for $|z| \leqq 1$ implies that the system is 0-stabilizable i.o.d.. If we generalize the notion of γ-stabilizability i.o.d. to allow for feedback matrices over $\mathbb{R}_0(z)$, it follows directly from Sontag's results [7, p. 24] that the system is γ-stabilizable i.o.d. for <u>any</u> $\gamma \geqq 0$ if $\operatorname{rank} U(z) = n$ for $|z| \leqq 1$. However, in this work we are requiring that the feedback matrix be polynomial, and as a consequence, $\operatorname{rank} U(z) = n$ for $|z| \leqq 1$ is not sufficient in general to insure

γ-stabilizability i.o.d. for any $\gamma > 0$.

Example 1: Suppose that $m = n = 1$ and $F(z) = 1$, $G(z) = z - a$, where a is any real number with $|a| > 1$. Then $U(z) = G(z) = z - a$, and thus rank $U(z) = 1$ for $|z| \leqq 1$. By Theorem 2, the system is γ-stabilizable i.o.d. if and only if there is a polynomial $L(z)$ such that $\mathrm{Re}(F(z) - G(z)L(z)) < -\gamma$ for $|z| \leqq e^{h\gamma}$. But since $F(a) - G(a)L(a) = 1$, the system is not γ-stabilizable i.o.d. for any value of γ for which $|a| \leqq e^{h\gamma}$.

It is very easy to construct an example which shows that rank $U(z) = n$ for $|z| \leqq 1$ is also not necessary in general for γ-stabilizability i.o.d..

Example 2: Suppose that $m = n = 1$ and $F(z) = z - .5$, $G(z) = z$. Note that since $F(1) = .5 > 0$, the system is not 0-stable i.o.d.. Now $U(z) = z$, and thus the rank of $U(z)$ is not equal to one when $z = 0$. But the system is obviously γ-stabilizable i.o.d. for any $\gamma < .5$: Taking $L(z) = 1$, we have that

$$F(z) - G(z)L(z) = -.5, \text{ all } z \in \mathbb{C}.$$

We shall conclude this section with the following characterization of the rank condition (4.3).

Proposition 1: For any fixed $\gamma \geqq 0$, the rank condition (4.3) is equivalent to

$$\mathrm{rank}[sI - F(z) \quad G(z)] = n, \; s \in \mathbb{C}, \; |z| \leqq e^{h\gamma}. \tag{4.8}$$

Proof: The result follows by applying point-by-point the Hautus' reachability criterion [28] generalized to systems over $\mathbb{C}$.

In the next section we will show that a weaker version of the rank condition (4.8) is necessary, but not sufficient, for γ-stabilizability i.o.d..

V. LOCAL STABILIZABILITY

For any fixed $z \in \mathbb{C}$, $F(z)$, $G(z)$, and $H(z)$ are matrices over the field of complex numbers, and thus for each $z \in \mathbb{C}$ the triple $(F(z),G(z),H(z))$ defines a linear finite-dimensional (delay-free) system over $\mathbb{C}$. In the study of stabilizability independent of delay, we can first ask whether or not these (local) systems are stabilizable as systems over $\mathbb{C}$. This leads to the following concept.

Definition 4: Given a fixed $\gamma \geqq 0$, the system defined by (2.2) is locally γ-stabilizable if and only if for every $z \in \mathbb{C}$ with $|z| \leqq e^{h\gamma}$, there is a $m \times n$ matrix L_z over $\mathbb{C}$ (or $\mathbb{R}$) such that

$$\operatorname{Re} \lambda_j(F(z) - G(z)L_z) < -\gamma, \quad j = 1,2,\ldots,n.$$

By Theorem 2, we see that local γ-stabilizability is necessary for γ-stabilizability i.o.d.. One might conjecture that local γ-stabilizability is also sufficient for γ-stabilizability i.o.d.. The question as to whether or not local stabilizability implies "global" stabilizability also arises in the study of systems depending on parameters [11],[12] and in the study of discrete-time systems defined over a commutative normed algebra [29],[30]. Unfortunately, for the particular framework considered here, local stabilizability is not sufficient in general for stabilizability i.o.d..

Example 3: Suppose that $m = n = 1$, $F(z) = z - a$, where $0 < a < .5$, and $G(z) = z^2$. Let $\gamma = 0$. Then setting

$$L_z = \begin{cases} 1/z, & |z| \leqq 1, \text{ except when } z = 0 \\ 0, & \text{when } z = 0 \end{cases}$$

we have that

$$F(z) - G(z)L_z = -a, \quad |z| \leqq 1.$$

Since $a > 0$, the system is locally 0-stabilizable. Now suppose that the system is 0-stabilizable i.o.d.; that is, there is a polynomial $L(z)$ such that

$$\mathrm{Re}(F(z) - G(z)L(z)) < 0, \quad |z| \leqq 1. \tag{5.1}$$

If (5.1) holds, there must exist real numbers $a_2, a_3, \ldots, a_N$ such that

$$\cos\omega \; - a + \sum_{j=2}^{N} a_j \cos j\omega \; < 0, \; \omega \in [0,2\pi]. \tag{5.2}$$

Letting $f(\omega)$ denote the left-hand side of the inequality (5.2) and using the expression for the coefficient of $\cos\omega$ of a Fourier series, we have that

$$1 = \frac{1}{\pi}\int_0^{2\pi} f(\omega)\cos\omega \, d\omega \leqq \frac{1}{\pi}\int_0^{2\pi} |f(\omega)| d\omega. \tag{5.3}$$

Using the expression for the constant term of a Fourier series, we also have that

$$-a = \frac{1}{2\pi}\int_0^{2\pi} f(\omega) d\omega.$$

But by (5.2), $f(\omega) < 0$ for $\omega \in [0,2\pi]$, and thus

$$a = \frac{1}{2\pi}\int_0^{2\pi} |f(\omega)| d\omega.$$

Then using (5.3), we get that $a \geqq .5$, a contradiction. Hence the system is not 0-stabilizable i.o.d..

In some interesting cases, local γ-stabilizability is equivalent to γ-stabilizability i.o.d..

Example 4: Suppose that $m = n = 1$, $F(z)$ = an arbitrary polynomial in z, and $G(z) = az + b$, $a \neq 0$. Given a fixed $\gamma \geqq 0$, we assume that $|b/a| \leqq e^{h\gamma}$. Then $U(z) = G(z)$ has a zero at $z = -b/a$, and by the assumption on $|b/a|$, we see that the rank condition (4.3) is not satisfied. Now since the only zero of $U(z)$ is at $z = -b/a$, it follows that the system is locally γ-stabilizable if and only if $F(-b/a) < -\gamma$. Let's suppose that this is the case. Dividing $G(z)$ into $F(z)$, we get

$$\frac{F(z)}{G(z)} = Q(z) + \frac{c}{G(z)} , \tag{5.4}$$

where $c = F(-b/a)$ and $Q(z)$ is a polynomial in z whose degree is less than that of $F(z)$ when $F(z) \neq$ constant. Rewriting (5.4), we have that

$$F(z) - G(z)Q(z) = F(-b/a) < -\gamma.$$

Hence with $L(z) = Q(z)$, the resulting closed-loop system is γ-stable i.o.d..

By applying point-by-point the Hautus' stabilizability criterion [31] generalized to finite-dimensional systems over $\mathbb{C}$, we get the following rank condition for local γ-stabilizability.

Proposition 2: The system $(F(z),G(z),H(z))$ defined by (2.2) is locally γ-stabilizable if and only if

$$\operatorname{rank}[sI - F(z) \quad G(z)] = n, \ \operatorname{Re} s \geqq -\gamma, \ |z| \leqq e^{h\gamma}. \tag{5.5}$$

By Proposition 1,

$$\operatorname{rank}[sI - F(z) \quad G(z)] = n, \ s \in \mathbb{C}, \ |z| \leqq e^{h\gamma} \tag{5.6}$$

is sufficient (but not necessary) for γ-stabilizability i.o.d.. Combining this fact with the above results, we see that any necessary and sufficient condition for γ-stabilizability i.o.d. must "sit between" the rank conditions (5.5) and (5.6). However, as the following example shows, there is no necessary and sufficient condition specified completely in terms of the rank of $[sI - F(z) \quad G(z)]$.

Example 5: Again let $m = n = 1$ and consider the systems $(z - a, z^2, 1)$ and $(z - a, z, 1)$, where $0 < a < .5$. It was shown in Example 3 that $(z - a, z^2, 1)$ is not 0-stabilizable i.o.d.. On the other hand, taking $L(z) = 1$, we see that $(z - a, z, 1)$ is 0-stabilizable i.o.d.. But

$$\text{rank}[s - z + a \quad z^2] = \text{rank}[s - z + a \quad z] \text{ for all } s, z \in \mathbb{C},$$

and thus in terms of the rank of $[sI - F(z) \quad G(z)]$, there is no difference between these two systems.

It is interesting to compare the above results with those of Pandolfi [32]. It follows as a special case of [32] that if $G(z)$ is delay free (i.e., $G(z)$ is over $\mathbb{R}$) and if distributed delays are allowed in the feedback, then

$$\text{rank}[sI - F(e^{-hs}) \quad G] = n, \ \text{Re}\, s \geqq 0$$

is necessary and sufficient for stabilizability (for a fixed delay h). Given this result, one might expect that when $\gamma = 0$ the rank condition (5.5) is necessary and sufficient for 0-stabilizability. But by Example 3 we know that this is not the case in general. Part of the reason for this negative result is that we are not allowing distributed delays in the feedback. Distributed delays can often be implemented using pure delays and integrators, and thus the use of distributed delays actually corresponds to dynamic feedback.

Although the rank condition (5.5) is not sufficient for γ-stabilizability i.o.d., it is necessary, and hence one could first check to see if (5.5) is satisfied. Unfortunately, (5.5) is not easy to test for as a consequence of the need to consider all values of z for which $|z| \leqq e^{h\gamma}$. In particular, it is not sufficient in general to consider the rank of $[sI - F(z) \quad G(z)]$ for $|z| = e^{h\gamma}$. Another problem from a practical standpoint is that testing for (5.5) would not tell us how to compute a stabilizing feedback, assuming one exists. In the next section we deal with the problem of determining a stabilizing feedback.

VI. CONSTRUCTION OF A STABILIZING FEEDBACK

If rank $U(z) = n$ for all $z \in C$, in which case the system is γ-stabilizable i.o.d. for any $\gamma \geqq 0$, we can compute a stabilizing feedback using the constructive procedure sketched by Morse [6]. By Sontag's results [7, p. 24], if rank $U(z) = n$ for $|z| \leqq 1$ Morse's procedure can still be applied to yield a stabilizing feedback over $\mathbb{R}_0(z)$. But as noted in Section IV, rank $U(z) = n$ for $|z| \leqq 1$ is not necessary in general for γ-stabilizability i.o.d.. In the first part of this section, we give a stepwise procedure for computing a stabilizing feedback based on a necessary condition for γ-stabilizability i.o.d.. A second approach to the construction of a feedback is considered in the last part of this section.

Let $\gamma \geqq 0$ be fixed. Given the system $(F(z), G(z), H(z))$, select a grid $0 \leqq \omega_1 < \omega_2 < \cdots < \omega_N \leqq \pi$ of the interval $[0, \pi]$. The points in the grid could be evenly spaced, although this is not necessary. We could start by choosing a two-point grid with $\omega_1 = 0$ and $\omega_2 = \pi$.

If the system is γ-stabilizable i.o.d., for each point ω_k in the grid, there must exist matrices L_k and L_k' over the reals such that

$$\operatorname{Re} \lambda_j (F(e^{h\gamma + i\omega_k}) - G(e^{h\gamma + i\omega_k})(L_k + iL_k')) < -\gamma, \quad j = 1, 2, \ldots, n.$$

Note that the stabilizing matrix $L_k + iL_k'$ may be complex (i.e. $L_k' \neq 0$) since the entries of $F(e^{h\gamma + i\omega_k})$ and $G(e^{h\gamma + i\omega_k})$ are complex numbers in general. A stabilizing feedback $L_k + iL_k'$ can be computed using existing methods for stabilizing finite-dimensional (delay-free) systems over $\mathbb{R}$ generalized to systems over $\mathbb{C}$. If the real parts of the eigenvalues of $F(e^{h\gamma + i\omega_k})$ are already less than $-\gamma$, we can of course take $L_k = L_k' = 0$.

Once the L_k and L_k' have been calculated for $k = 1,2,\ldots,N$, we can then interpolate to a polynomial matrix $L(z)$ with real coefficients such that

$$\begin{aligned} \operatorname{Re} L(e^{h\gamma + i\omega_k}) &= L_k, \quad k = 1,2,\ldots,N, \\ \operatorname{Imag} L(e^{h\gamma + i\omega_k}) &= L_k'. \quad k = 1,2,\ldots,N. \end{aligned} \tag{6.1}$$

There is always a polynomial matrix with real coefficients satisfying (6.1) with degree less than or equal to 2N. For example, if we start with a two-point grid with $\omega_1 = 0$ and $\omega_2 = \pi$, since e^{i0} and $e^{i\pi}$ are real numbers, the stabilizing feedback matrices will be real, and a minimum-degree polynomial matrix satisfying (6.1) is given by

$$L(z) = .5e^{-h\gamma}(L_1 - L_2)z + .5(L_1 + L_2).$$

For any polynomial matrix $L(z)$ satisfying (6.1), by the continuity property mentioned in Section III, there exist open neighborhoods $V_k \subset \mathbb{R}$ of the ω_k such that

$$\operatorname{Re} \lambda_j(F(e^{h\gamma + i\omega}) - G(e^{h\gamma + i\omega})L(e^{h\gamma + i\omega})) < -\gamma, \; j = 1,2,\ldots,n, \; \omega \in \bigcup_{k=1}^{N} V_k.$$

If

$$[0,\pi] \subset \bigcup_{k=1}^{N} V_k, \tag{6.2}$$

by the corollary to Theorem 2, the closed-loop system with feedback matrix equal to $L(z)$ is γ-stable i.o.d.., and we are done. We can determine whether or not (6.2) is satisfied by applying the stability tests described in Section III.

If (6.2) is not satisfied, we can then consider a finer grid for $[0,\pi]$ containing the points in the first grid. For each point in this second grid not contained in the first grid, we construct a stabilizing feedback $L_k + iL_k'$. With these feedback matrices and the ones computed with respect to the first grid, we interpolate to a polynomial matrix $L(z)$ which satisfies (6.1) at all points of the new grid. If the closed-loop system with feedback matrix $L(z)$ is γ-stable i.o.d., we are done. If not, we can repeat the process until a solution is obtained or until the degree of the interpolating polynomial matrix becomes too large, in which case the procedure yields no solution.

It may not be true that the above procedure always converges to a stabilizing feedback whenever one exists, but we have found that the technique works well on simple examples. We plan to consider a computer implementation of the procedure which can be applied to examples for which the size of the system matrix $F(z)$ is not small.

For some classes of delay differential systems, it is possible to approach the construction of a stabilizing feedback by using results on the location of the eigenvalues of a matrix sum $A+B$ defined over $\mathbb{R}$ or $\mathbb{C}$, with the eigenvalues of A or B known a priori. One such result for the class of systems with a delay-free input matrix is given below. In the following result, the norm $\|Q\|$ of a $n \times n$ matrix Q over $\mathbb{C}$ is any matrix norm induced by an absolute vector norm on $\mathbb{C}^n$ [27, pp. 213-215].

Theorem 6: Suppose that

$$F(z) = F_0 + F_1 z + \cdots + F_q z^q \text{ and } G(z) = G,$$

where the F_k are $n \times n$ matrices over $\mathbb{R}$ and G is a $n \times m$ matrix over $\mathbb{R}$. Let $L_1, L_2, \ldots, L_q$ be $m \times n$ matrices over $\mathbb{R}$ such that for $k = 1,2,\ldots,q$, $\|F_k - GL_k\|$ is equal to or near the minimum possible value (with L_k ranging over all $m \times n$ matrices over $\mathbb{R}$). Suppose that there is a $m \times n$ matrix L_0 over $\mathbb{R}$ such that the eigenvalues $d_1, d_2, \ldots, d_n$ of $F_0 - GL_0$ are distinct and have negative real parts, and such that

$$\frac{\min_k |d_k|}{\nu(F_0 - GL_0)} > \sum_{k=1}^{q} \|F_k - GL_k\|, \tag{6.3}$$

where $\nu(F_0 - GL_0)$ is the condition number of $F_0 - GL_0$ [27, p. 232]. Then the closed-loop system with feedback matrix $L(z) = L_0 + L_1 z + \cdots + L_q z^q$ is 0-stable i.o.d..

Proof: Let $L_0, L_1, \ldots, L_q$ satisfy the hypothesis of the theorem. Since the eigenvalues of $F_0 - GL_0$ are distinct, $F_0 - GL_0$ is a simple matrix, and thus by a result in [27, p. 234], for any fixed z with $|z| \leq 1$ the eigenvalues of $F(z) - GL(z)$ lie in at least one of the discs

$$|s - d_k| \leq \left\| \sum_{j=1}^{q} (F_j - GL_j) z^j \right\| \nu(F_0 - GL_0), \quad k = 1,2,\ldots,n.$$

Now

$$\left\| \sum_{j=1}^{q} (F_j - GL_j) z^j \right\| \leq \sum_{j=1}^{q} \|F_j - GL_j\|, \quad |z| \leq 1,$$

and using (6.3), we have that for any z with $|z| \leqq 1$, the eigenvalues of $F(z) - GL(z)$ lie in at least one of the discs

$$|s - d_k| < \min_j |d_j|, \; k = 1,2,\ldots,n.$$

Therefore, by Theorem 2 the closed-loop system is 0-stable i.o.d..

If the pair (F_0,G) is reachable; that is, the rank of $[G \;\; F_0G \;\; \cdots \;\; F_0^{n-1}G]$ is equal to n, then by selecting L_0 we can place the eigenvalues of $F_0 - GL_0$ as far over in the left-half plane as we desire. Hence one might conclude that (6.3) can always be satisfied when (F_0,G) is reachable. However this is not the case, because as we move the eigenvalues of $F_0 - GL_0$ farther over in the left-half plane, in general the condition number $\nu(F_0 - GL_0)$ increases so that the left side of the inequality (6.3) can not be made arbitrarily large. Nevertheless, the existence of matrices $L_0,L_1,\ldots,L_q$ for which (6.3) is satisfied is not an unreasonable constraint, and thus the result in Theorem 6 is of interest.

VII. AN OBSERVER AND REGULATOR

If the state x(t) of the system (F(z),G(z),H(z)) is not directly accessible, so that state feedback is not possible, a common procedure is to construct an observer and then feed back the observer output. We shall sketch this process using a standard type of observer with the requirement that the observer error dynamics and the resulting closed-loop system be γ-stable independent of delay. For a survey of various existing notions of observability and observers for systems with time delays, the reader should refer to the paper of Lee and Olbrot [33].

A special case of a Luenberger-type observer for the system $(F(z),G(z),H(z))$ is given by the dynamical equations

$$\frac{d\xi(t)}{dt} = \sum_{k=0}^{q} F_k \xi(t-kh) + \sum_{j=0}^{\ell} W_j \left[y(t-jh) - \sum_{k=0}^{\mu} H_k \xi(t-jh-kh) \right] + \sum_{k=0}^{r} G_k u(t-kh). \tag{7.1}$$

In (7.1), $u(t)$ (resp., $y(t)$) is the input (output) of the given system, $\xi(t) \in \mathbb{R}^n$ is the instantaneous state of the observer, and $W_0, W_1, \ldots, W_\ell$ are the observer gain matrices. Letting $e(t) = x(t) - \xi(t)$ denote the error, we have the following equation for the error dynamics

$$\frac{de(t)}{dt} = \sum_{k=0}^{q} F_k e(t-kh) - \sum_{j=0}^{\ell} \sum_{k=0}^{\mu} W_j H_k e(t-jh-kh). \tag{7.2}$$

The characteristic function $P_e(s,e^{-hs})$ associated with the error equation (7.2) is given by

$$P_e(s,e^{-hs}) = \det(sI - F(z) + W(z)H(z)) \Big|_{z=e^{-hs}},$$

where $W(z) = \sum_{k=0}^{\ell} W_k z^k$.

Definition 5: Given a fixed $\gamma \geqq 0$, the observer (7.1) is said to be a γ-stable i.o.d. observer if and only if

$$P_e(s,e^{-as}) \neq 0, \ \mathrm{Re}\, s \geqq -\gamma, \ \text{all real numbers } a \geqq 0.$$

By Theorem 1, in a γ-stable i.o.d. observer the error process given by (7.2) is exponentially stable independent of delay with order $> \gamma$.

As is the case for finite-dimensional systems, one can approach the study of γ-stable i.o.d. observers by considering the dual system $(F^T(z), H^T(z), G^T(z))$, where T denotes the transpose operation.

Proposition 3: The system $(F(z),G(z),H(z))$ has a γ-stable i.o.d. observer given by (7.1) if and only if the dual system $(F^T(z),H^T(z),G^T(z))$ is γ-stabilizable i.o.d.. Further, if the closed-loop system $(F^T(z) - H^T(z)L(z), H^T(z),G^T(z))$ is γ-stable i.o.d., then with $W(z) = L^T(z)$, (7.1) is a γ-stable i.o.d. observer for $(F(z),G(z),H(z))$.

An obvious consequence of Proposition 3 is that all of the results on state feedback in Sections IV-VI can be carried over to the observer framework defined here.

Once we have constructed a γ-stable i.o.d. observer, we can then feed back the state $\xi(t)$ of the observer by setting

$$u(t) = -\sum_{k=0}^{\eta} L_k \xi(t-kh). \tag{7.3}$$

It is easy to verify that the characteristic function $P_{cl}(s,e^{-hs})$ of the resulting closed-loop system is given by

$$P_{cl}(s,e^{-hs}) = \det(sI - F(e^{-hs}) + G(e^{-hs})L(e^{-hs}))\det(sI - F(e^{-hs}) + W(e^{-hs})H(e^{-hs}))$$

where $L(e^{-hs}) = \sum_{k=0}^{\eta} L_k e^{-khs}$.

The observer (7.1) with the feedback (7.3) is called a regulator. It is a direct consequence of the above expression for $P_{cl}(s,e^{-hs})$ that if $(F(z),G(z), H(z))$ and its dual are γ-stabilizable i.o.d., there is a regulator such that the closed-loop system is γ-stable i.o.d..

VIII. CONCLUDING REMARKS

In this work we restricted attention to nondynamic state feedback and to a special case of a Luenberger-type observer. We have shown that local γ-stabilizability (resp., local γ-stabilizability of the dual system) is necessary but not sufficient for the existence of a nondynamic stabilizing feedback (resp., a standard type of observer). An interesting open question is whether or not local γ-stabilizability (resp., local γ-stabilizability of the dual) is sufficient for the existence of a dynamic stabilizing feedback (resp., a more general type of observer). It may be possible to answer this question by adapting the constructions of Hautus and Sontag [14] and Khargonekar and Emre [20].[1] Even if there is an equivalence between local stabilizability and global stabilizability using dynamic feedback, nondynamic feedback is still of interest because of its relative simplicity. In particular, as shown in Section VI, it is possible to approach the construction of a nondynamic stabilizing feedback using results for stabilizing finite-dimensional systems.

It should also be mentioned that by working with polynomials in several variables (see [10]), we can generalize the approach in this paper to systems with noncommensurate time delays. We can even consider delay differential systems of the neutral type with noncommensurate delays. In fact, it follows directly from the recent work of Guiver and Bose [35] that there is a generalization of Theorem 2 for a large class of delay differential systems of the neutral type with noncommensurate delays.

[1] Sontag has informed the author that results along this line are in a paper under preparation [34].

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